Fair and Efficient Allocations under Lexicographic Preferences

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Abstract

Envy-freeness up to any good (EFX) provides a strong and intuitive guarantee of fairness in the allocation of indivisible goods. But whether such allocations always exist or whether they can be efficiently computed remains an important open question. We study the existence and computation of EFX in conjunction with various other economic properties under lexicographic preferences—a well-studied preference model in artificial intelligence and economics. In sharp contrast to the known results for additive valuations, we not only prove the existence of EFX and Pareto optimal allocations, but in fact provide an algorithmic characterization of these two properties. We also characterize the mechanisms that are, in addition, strategyproof, non-bossy, and neutral. When the efficiency notion is strengthened to rank-maximality, we obtain non-existence and computational hardness results, and show that tractability can be restored when EFX is relaxed to another well-studied fairness notion called maximin share guarantee (MMS).

Introduction

Fair and efficient allocation of scarce resources is a fundamental problem in economics and computer science. The quintessential fairness notion—envy-freeness—enjoys strong existential and computational guarantees for divisible resources (Varian 1974). However, in notable applications such as course allocation (Budish 2011) and property division (Prush and Woeginger 2012) that involve indivisible resources, (exact) envy-freeness could be too restrictive. In these settings, it is natural to consider notions of approximate fairness such as envy-freeness up to any good (EFX) wherein pairwise envy can be eliminated by the removal of any good in the envied bundle (Caragiannis et al. 2019).

EFX is arguably the closest analog of envy-freeness in the indivisible setting, and, as a result, has been actively studied especially for the domain of additive valuations. However, it also suffers from a number of limitations: First, barring a few special cases, the existence and computation of EFX allocations remains an open problem. Second, for additive valuations, EFX can be incompatible with Pareto optimality (PO)—a fundamental notion of economic efficiency (Plaut and Roughgarden 2020). Finally, EFX could also be at odds with strategyproofness (Amanatidis et al. 2017), which is another desirable property in the economic analysis of allocation problems.

The aforementioned limitations of EFX prompt us to explore the domain restriction approach in search of positive results (Elkind, Lackner, and Peters 2016). Specifically, we deviate from the framework of cardinal preferences for which EFX allocations have been most extensively studied, and instead focus on the purely ordinal domain of lexicographic preferences.

Lexicographic preferences have been widely studied in psychology (Gigerenzer and Goldstein 1996), machine learning (Schmitt and Martignon 2006), and social choice (Taylor 1970) as a model of human decision-making. Several real-world settings such as evaluating job candidates and the desirability of a product involve lexicographic preferences over the set of features. In the context of fair division, too, lexicographic preferences can arise naturally. For example, when dividing an inheritance consisting of a house, a car, and some home appliances, a stakeholder might prefer any division in which she gets the house over one where she doesn’t (possibly because of its sentimental value), subject to which she might prefer any outcome that includes the car over one that doesn’t, and so on.

On the computational side, lexicographic preferences provide a succinct language for representing preferences over combinatorial domains (Saban and Sethuraman 2014; Lang, Megen, and Xiu 2018), and have led to numerous positive results at the intersection of artificial intelligence and economics (Fujita et al. 2018; Hosseini and Larson 2019). Motivated by these considerations, our work examines the existence and computation of fair (i.e., EFX) and efficient allocations from the lens of lexicographic preferences.

Our Contributions. Figure 1 summarizes our theoretical contributions.

- \textbf{EFX+PO}: Our first result provides a family of polynomial-time algorithms for computing EFX+PO allocations under lexicographic preferences. Furthermore, we show that any EFX+PO allocation can be computed by some algorithm in this family, thus providing an algorithmic characterization of such allocations (Theorem 1). This result establishes a sharp contrast with the additive valuations domain where the two properties are incompatible in general.

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EFX+PO+strategyproofness: The positive result for EFX+PO motivates us to investigate a more demanding property combination of EFX, PO, and strategyproofness. Once again, we obtain an algorithmic characterization (Theorem 2). Subject to some common axioms (non-bossiness and neutrality), any mechanism satisfying EFX, PO, and strategyproofness is characterized by a special class of quota-based serial dictatorship mechanisms (Pápai 2000b; Hosseini and Larson 2019).

EFX+rank-maximality: When the efficiency notion is strengthened to rank-maximality, we encounter incompatibility with strategyproofness (Example 1) as well as with EFX (Example 2). Furthermore, checking the existence of EFX and rank-maximal allocations turns out to be NP-complete (Theorem 3), suggesting that our algorithmic results are, in a certain sense, 'maximal'. The intractability persists even when EFX is relaxed to envy-freeness up to $k$ goods (EF$k$) (Theorem 4), but efficient computation is possible if EFX is relaxed to another well-studied fairness notion called maximin share guarantee or MMS (Theorem 5).

Related Work. Envy-free solutions may not always exist for indivisible goods. As a result, the literature has focused on notions of approximate fairness, most notably envy-freeness up to one good (EF1) and its strengthening called envy-freeness up to any good (EFX). The former enjoys strong existential and algorithmic support, as an EF1 allocation always exists for general monotone valuations and can be efficiently computed. However, achieving EF1 together with economic efficiency seems non-trivial: For additive valuations, EF1+PO allocations always exist (Caragiannis et al. 2019; Barman, Krishnamurthy, and Vaish 2018) but no polynomial-time algorithm is known for computing such allocations.

The stronger notion of EFX has proven to be more challenging. As mentioned previously, the existence of EFX for additive valuations remains an open problem. Additionally, EFX and Pareto optimality are known to be incompatible for non-negative additive valuations (Plaut and Roughgarden 2020).

The aforementioned limitations of EFX have motivated the study of further relaxations or special cases in search of positive results. Some recent results establish the existence of partial allocations that satisfy EFX after discarding a small number of goods while also fulfilling certain efficiency criteria (Caragiannis, Gravin, and Huang 2019; Chaudhury et al. 2020). Similarly, EFX allocations have been shown to exist for the special case of three agents with additive valuations (Chaudhury, Garg, and Mehlhorn 2020), or when the agents can be partitioned into two types (Mahara 2020), or when agents have dichotomous preferences (Amanatidis et al. 2020). For cardinal utilities, various multiplicative approximations of EFX (and its variant that involves removing an average good) have been considered (Plaut and Roughgarden 2020; Amanatidis, Markakis, and Ntokos 2020; Chaudhury, Garg, and Mehta 2020). Another emerging line of work studies EFX for non-monotone valuations, i.e., when the resources consist of both goods and chores (Chen and Liu 2020; Bérczi et al. 2020).

The interaction between fairness and efficiency is further complicated with the addition of strategyproofness due to several fundamental impossibility results both in deterministic (Zhou 1990) as well as randomized settings (Bogomolnaia and Moulin 2001; Kojima 2009). Indeed, while ordinal efficiency is compatible with envy-freeness, such outcomes cannot, in general, be achieved via (weakly) strategyproof mechanisms even under strict preferences (Kojima 2009). Moreover, sd-efficiency and sd-strategyproofness (here, sd stands for stochastic dominance) are incompatible even with a weak notion of stochastic fairness called equal treatment of equals (Aziz and Kasajima 2017). In a similar vein, for deterministic mechanisms, any strategyproof mechanism could fail to satisfy EFX even for two agents under additive valuations (Amanatidis et al. 2017).

Lexicographic preferences have also been successfully used as a domain restriction to circumvent impossibility results in mechanism design (Sikdar, Adali, and Xia 2017; Fujita et al. 2018). In fair division of indivisible goods, lexicographic (sub)additive utilities have facilitated constant-factor approximation algorithms for egalitarian and Nash social welfare objectives (Baumeister et al. 2017; Nguyen 2020). Hosseini and Larson (2019) show that under lexicographic preferences, a mechanism is Pareto optimal, strategyproof, non-bossy, and neutral if and only if it is a serial dictatorship quota mechanism. In randomized settings, too, lexicographic preferences have led to the design of mechanisms that simultaneously satisfy stochastic efficiency, envy-freeness, and strategyproofness (Schulman and Vazirani 2015; Hosseini and Larson 2019).

Preliminaries

Model. For any $k \in \mathbb{N}$, define $[k] := \{1, \ldots, k\}$. An instance of the allocation problem is a tuple $(N, M, \succ)$, where $N := \{n\}$ is a set of $n$ agents, $M$ is a set of $m$ goods, and $\succ := (\succ_1, \ldots, \succ_n)$ is a preference profile that specifies the ordinal preference of each agent $i \in N$ as a linear order $\succ_i \in \mathcal{L}$ over the set of goods; here, $\mathcal{L}$ denotes the set of all (strict and complete) linear orders over $M$.

Allocation and bundles. A bundle is any subset $X \subseteq M$ of the set of goods. An allocation $A = (A_1, \ldots, A_n)$ is an $n$-partition of $M$, where $A_i \subseteq M$ is the bundle assigned to agent $i$. We will write $A_i$ to denote the set of all $n$-partitions of $M$. We say that allocation $A$ is partial if $\bigcup_{i \in N} A_i \subset M$, and complete if $\bigcup_{i \in N} A_i = M$.

Lexicographic preferences. We will assume that agents’ preferences over the bundles are given by the lexicographic
extension of their preferences over individual goods. Informally, this means that if an agent ranks the goods in the order $a > b > c > \ldots$, then it prefers a bundle containing $a$ over any other bundle that doesn’t, subject to that, it prefers a bundle containing $b$ over any other bundle that doesn’t, and so on. Formally, given any pair of bundles $X, Y \subseteq M$ and any linear order $\succ_i \in L$, we have $X \succ_i Y$ if and only if there exists a good $g \in X \setminus Y$ such that $g' \in Y : g' \succ_i g \subseteq X$.

Notice that since $\succ_i$ is a linear order over $M$, the corresponding lexicographic extension is a linear order over $2^M$.

For any agent $i \in N$ and any pair of bundles $X, Y \subseteq M$, we will write $X \succeq_i Y$ if either $X \succ_i Y$ or $X = Y$.

**Envy-freeness** Given a preference profile $\succ$, an allocation $A$ is said to be (a) envy-free (EF) if for every pair of agents $i, h \in N$, we have $A_i \succeq_i A_h$, (b) envy-free up to any good (EFX) if for every pair of agents $i, h \in N$ such that $A_i \neq \emptyset$ and every good $j \in A_h$, we have $A_i \succeq_i A_h \setminus \{j\}$, and (c) envy-free up to $k$ goods (EF$k$) if for every pair of agents $i, h \in N$ such that $A_i \neq \emptyset$, there exists a set $S \subseteq A_h$ such that $|S| \leq k$ and $A_i \succeq_i A_h \setminus S$. Clearly, EFX $\Rightarrow$ EF $\Rightarrow$ EF2 $\Rightarrow \ldots$.

**Maximin Share** An agent’s maximin share is its most preferred bundle that it can guarantee itself as a divider in an $n$-person cut-and-choose procedure against adversarial opponents (Budish, 2011). Formally, the maximin share of agent $i$ is given by $\text{MMS}_i := \max_{A \in \Pi} \min_i \{A_1, \ldots, A_n\}$, where $\min_i \{\} \text{ and } \max_i \{\}$ denote the least-preferred and most-preferred bundles with respect to $\succ_i$. An allocation $A$ satisfies maximin share guarantee (MMS) if each agent receives a bundle that it weakly prefers to its maximin share. That is, the allocation $A$ is MMS if for every $i \in N$, $A_i \succeq_i \text{MMS}_i$. It is easy to see that EF $\Rightarrow$ MMS. Additionally, for lexicographic preferences, we have that EFX $\Rightarrow$ MMS (the converse is not true) while EF1 and MMS can be incomparable (see the full version [Hosseini et al., 2020] for details).

**Pareto optimality** Given a preference profile $\succ$, an allocation $A$ is said to be Pareto optimal (PO) if there is no other allocation $B$ such that $B_i \succeq_i A_i$ for every agent $i \in N$ and $B_k \succ_k A_k$ for some agent $k \in N$.

**Rank-maximality** A rank-maximal (RM) allocation is one that maximizes the number of agents who receive their favorite good, subject to which it maximizes the number of agents that receive their second favorite good, and so on ([Irving et al., 2006] [Paluch, 2013]). Given an allocation $A$, its signature refers to a tuple $(n_1, n_2, \ldots, n_m)$ where $n_i$ is the number of agents who receive their $i$th favorite good (note that an agent can contribute to multiple $n_i$'s). All rank-maximal allocations for a given instance have the same signature. Computing some rank-maximal allocation for a given instance is easy: Assign each good to an agent that ranks it the highest among all agents (tiebreak arbitrarily). This procedure provides a computationally efficient way of computing the signature of a rank-maximal allocation as well as verifying whether a given allocation is rank-maximal. Notice that rank-maximality is a strictly stronger requirement than Pareto optimality.

**Mechanism** A mechanism $f : L^n \rightarrow \Pi$ is a mapping from preference profiles to allocations. For any preference profile $\succ \in L^n$, we use $f(\succ)$ to denote the allocation returned by $f$, and $f_i(\succ)$ to denote the bundle assigned to agent $i$.

**Properties of mechanisms** A mechanism $f : L^n \rightarrow \Pi$ is said to satisfy EF / EFX / EF$k$ / PO / RM if for every preference profile $\succ \in L^n$, the allocation $f(\succ)$ has that property. In addition, a mechanism $f$ satisfies

- **strategyproofness** (SP) if no agent can improve by misreporting its preferences. That is, for every preference profile $\succ \in L^n$, every agent $i \in N$, and every (misreported) linear order $\succ'_i \in L$, we have $f_i(\succ \succ_i \succ'_i) \leq f_i(\succ)$, where $\succ' := (\succ_1, \ldots, \succ_{i-1}, \succ'_i, \succ_{i+1}, \ldots, \succ_n)$.
- **non-bossiness** if no agent can modify the allocation of another agent by misreporting its preferences without changing its own allocation. That is, for every preference profile $\succ \in L^n$, every agent $i \in N$, and every (misreported) linear order $\succ' \in L$, we have $f_i(\succ') \not\leq f(\succ)$, where $\succ' := (\succ_1, \ldots, \succ_{i-1}, \succ'_i, \succ_{i+1}, \ldots, \succ_n)$.
- **neutrality** if relabeling the goods results in a consistent change in the allocation. That is, for every preference profile $\succ \in L^n$ and every relabeling of the goods $\pi : M \rightarrow M$, it holds that $f(\pi(\succ)) = \pi(f(\succ))$, where $\pi(\succ') := (\pi(\succ_1), \ldots, \pi(\succ_n))$ and $\pi(A) := (\pi(A_1), \ldots, \pi(A_n))$ for any allocation $A = (A_1, \ldots, A_n)$.

**EFX and Pareto Optimality**

Recall that for additive valuations, establishing the existence of EFX allocations remains an open problem, and there exist instances where no allocation is simultaneously EFX and PO ([Plaut and Roughgarden, 2020]). Our first result (Theorem 1) shows that there is no conflict between fairness and efficiency for lexicographic preferences: Not only does there exist a family of polynomial-time algorithms that always return EFX+PO allocations, but every EFX+PO allocation can be computed by some algorithm in this family. We will start with an easy observation concerning EFX allocations.

**Proposition 1.** An allocation $A$ is EFX if and only if each envied agent in $A$ gets exactly one good.

**Description of algorithm** Each algorithm in this family (Algorithm 1) is specified by an ordering $\sigma$ over the agents, and consists of two phases. Phase 1 involves a single round of serial dictatorship according to $\sigma$. Phase 2 assigns the remaining goods among the unenvied agents according to a picking sequence $\tau$. Note that the set of unenvied agents after Phase 1 must be nonempty; in particular, the last agent in $\sigma$ belongs to this set since every other agent prefers its good picked in Phase 1 to any good in the last agent’s bundle.

**Theorem 1.** For any ordering $\sigma$ of the agents, the allocation computed by Algorithm 1 satisfies EFX and PO. Conversely, any EFX+PO allocation can be computed by Algorithm 1 for some choice of $\sigma$.

**Proof.** We will start by showing that the allocation $A$ returned by Algorithm 1 satisfies EFX. From Proposition 1, it suffices to show that any envied agent gets exactly one good in $A$. Notice that any agent that is envied at the end of Phase 1 does not receive any good in Phase 2. Furthermore, the pairwise envy relations remain unchanged during Phase 2 since each agent has already picked its favorite available
Parameters: A permutation $\sigma : N \to N$ of the agents
Output: An allocation $A$

ALGORITHM 2:
Input: An instance $\langle N, M, \succ \rangle$ with lexicographic preferences
Parameters: A permutation $\sigma : N \to N$ of the agents
Output: An allocation $A$

1. $A \leftarrow (\emptyset, \ldots, \emptyset)$
2. Execute one round of serial dictatorship according to $\sigma$.
3. Assign all remaining goods to the last agent in $\sigma$.
4. return $A$

Indeed, in any PO allocation, some agent must receive its favorite good (otherwise a cyclic exchange of the top-ranked goods gives a Pareto improvement). Add this agent to the picking sequence, and repeat the procedure for the remaining goods.

ditive valuations, strategyproofness is known to be incompatible even with EFX (Amanatidis et al. 2017). By contrast, for lexicographic preferences, we will show that strategyproofness can be achieved in conjunction with a stronger fairness guarantee (EFX) as well as Pareto optimality, non-bossiness, and neutrality (Theorem 2). Indeed, Algorithm 2 is a special case of Algorithm 1 where the last agent gets all the remaining goods characterizes these properties.

Our characterization result builds upon an existing result of Hosseini and Larson (2019) Theorem 5.6 (see Proposition 2) that characterizes four out of the five properties mentioned above (excluding EFX) in terms of Serial Dictatorship Quota Mechanisms (SDQ), as defined below.

Definition 1. The Serial Dictatorship Quota (SDQ) mechanism is specified by a permutation $\sigma : N \to N$ of the agents and a set of quotas $(q_1, \ldots, q_n)$ such that $\sum_{i=1}^n q_i = m$. Given a lexicographic instance $\langle N, M, \succ \rangle$ as input, the SDQ mechanism considers agents in the order $\sigma$, and assigns the $i$th agent its most preferred bundle of size $q_i$ from the remaining goods. The resulting allocation is returned as output.

Proposition 2 (Hosseini and Larson 2019). For lexicographic preferences, a mechanism is Pareto optimal, strategyproof, non-bossy, and neutral if and only if it is SDQ.

The next result (Theorem 2) provides an algorithmic characterization of EFX, PO, strategyproofness, non-bossiness, and neutrality for lexicographic preferences.

Theorem 2. For any ordering $\sigma$ of the agents, Algorithm 2 is EFX, PO, strategyproof, non-bossy, and neutral. Conversely, any mechanism satisfying these properties can be implemented by Algorithm 2 for some $\sigma$.

Proof. Note that Algorithm 2 is a special case of SDQ for the quotas $q_i = 1$ for all $i \in [n-1]$ and $q_n = m - (n-1)$. Therefore, from Proposition 2 it is PO, strategyproof, non-bossy, and neutral. Furthermore, Algorithm 2 is also a special case of Algorithm 1 and is therefore EFX (Theorem 1).

To prove the converse, let $f$ be an arbitrary mechanism satisfying the desired properties. From Proposition 2, $f$ must be an SDQ mechanism for some ordering $\sigma$ and some set of quotas $(q_1, \ldots, q_n)$ such that $\sum_{i=1}^n q_i = m$. If $m < n$, the claim follows easily from Theorem 1 so we can assume, without loss of generality, that $m \geq n$. Then, by Proposition 1 we must have that $q_i \geq 1$ for all $i \in [n]$. Therefore, it suffices to show that $q_i = 1$ for all $i \in [n-1]$.

Assume, without loss of generality, that $\sigma = (1, 2, \ldots, n)$. Consider a preference profile $\succ$ with identical preferences, i.e., $\succ_i = \succ_k$ for all $i, k \in [n]$. Let
Each of the goods \( g_1 \succ_i g_2 \succ_i \cdots \succ_i g_m \) for any \( i \in [n] \). Suppose, for contradiction, that \( g_i \succ 1 \) for some \( i \in [n-1] \), and let \( k \in [n-1] \) be the smallest index for which this happens. Since \( f \) is an SDQ mechanism, we have that \( g_k \in f_k(\succ) \) and \( |f_k(\succ)| = q_k > 1 \). Then, for every \( \ell > k \), agent \( \ell \) envies agent \( k \). By Proposition 1, \( f \) violates EFX, which is a contradiction. Therefore, \( f \) must be identical to Algorithm 2 for the ordering \( \sigma \), as desired.

We note that in this context any deterministic strategyproof and non-bossy mechanism is an agent group-strategyproof (Papai 2000a). Therefore, Algorithm 2 also characterizes the set of EFX, PO, group-strategyproof, non-bossy, and neutral mechanisms under lexicographic preferences. In addition, we show that the set of properties considered in Theorem 2 is minimal. That is, dropping any property from the characterization necessarily allows for feasible mechanisms beyond those in Algorithm 2. Details and the proof are relegated to the full version (Hosseini et al. 2020).

**Proposition 3.** The set \{EFX, PO, strategyproofness, non-bossiness, neutrality\} is a minimal set of properties for characterizing the family of mechanisms in Algorithm 2.

The efficiency guarantee in Theorem 2 cannot be strengthened further, as there exists an instance where any rank-maximal (RM) mechanism violates strategyproofness (Example 1).

**Example 1 (Strategyproofness and RM).** Consider the instance with \( k+2 \) goods \( g_1, \ldots, g_{k+2} \) and three agents:

- \( a_1 : g_1 \succ g_2 \succ g_3 \succ \cdots \succ g_{k+1} \succ g_{k+2} \)
- \( a_2 : g_1 \succ g_2 \succ g_3 \succ \cdots \succ g_{k+1} \succ g_{k+2} \)
- \( a_3 : g_2 \succ g_3 \succ g_4 \succ \cdots \succ g_{k+2} \succ g_1 \).

Each of the goods \( g_2, \ldots, g_{k+2} \) is ranked higher by \( a_3 \) than by \( a_1 \) or \( a_2 \), and therefore must be assigned to \( a_3 \) in any rank-maximal allocation. Suppose, under truthful reporting, \( g_1 \) is assigned to \( a_1 \), and \( a_2 \) gets an empty bundle. Then, \( a_2 \) could falsely report \( g_3 \) as its favorite good. By rank-maximality, \( g_3 \) is now assigned to \( a_2 \), resulting in a strict improvement.

The non-existence result in Example 1 prompts us to forego strategyproofness (as well as non-bossiness and neutrality) and focus only on (approximate) envy-freeness and rank-maximality.

**Envy-Freeness and Rank-Maximality**

For lexicographic preferences, it is easy to see that a complete allocation is envy-free if and only if each agent receives its favorite good. Checking the existence of an envy-free allocation therefore boils down to computing a \((\ell,\ell)-\)perfect matching in a bipartite graph where the left and the right vertex sets correspond to the agents and the goods, respectively, and the edges denote the top-ranked good of each agent. If an envy-free partial allocation of the top-ranked goods exists, then it can be extended to a complete rank-maximal allocation by assigning each remaining good to an agent that has the highest rank for it (note that the assignment of the remaining goods does not introduce any envy). Thus, the existence of an envy-free and rank-maximal allocation can be checked efficiently for lexicographic preferences (Proposition 4).

**Proposition 4.** There is a polynomial-time algorithm that, given a lexicographic instance as input, computes an envy-free and rank-maximal allocation, whenever one exists.

Since an envy-free allocation is not guaranteed to exist, one could ask whether rank-maximality can always be achieved alongside approximate envy-freeness; in particular, EFX and EFX. Example 2 shows that both of these notions could conflict with rank-maximality. Specifically, for any fixed \( k \in \mathbb{N} \), an EFX+-RM allocation could fail to exist. Since EFX implies EF1, a similar incompatibility holds for EFX+RM as well.

**Example 2 (EFk and RM).** Consider again the instance in Example 2. Any rank-maximal allocation assigns the goods \( g_2, \ldots, g_{k+2} \) to \( a_3 \). If \( g_1 \) is assigned to \( a_1 \), then \( a_2 \) gets an empty bundle, and the pair \( \{a_2, a_3\} \) violates EFX.

Given the non-existence result in Example 2, a natural question is whether there exists an efficient algorithm for checking the existence of an approximately envy-free and rank-maximal allocation. Unfortunately, the news here is also negative, as the problem turns out to be NP-complete (Theorem 3). Thus, while EFX can always be achieved in conjunction with Pareto optimality (Theorems 1 and 2), strengthening the efficiency notion to rank-maximality results in non-existence and computational hardness.

**Theorem 3.** Determining whether a given instance admits an EFX and rank-maximal allocation is NP-complete.

Proof. Membership in NP follows from the fact that both EFX and rank-maximality can be checked in polynomial time. To prove NP-hardness, we will show a reduction from a restricted version of 3-SAT called \((2/2/3)\)-SAT, which is known to be NP-complete (Ahadi and Dehghan 2019).

An instance of \((2/2/3)\)-SAT consists of a collection of \( r \) variables \( X_1, \ldots, X_r \) and \( s \) clauses \( C_1, \ldots, C_s \), where each clause is specified as a disjunction of three literals, and each variable occurs in exactly four clauses, twice negated and twice non-negated. The goal is to determine if there is a truth assignment that satisfies all clauses.

**Construction of the reduced instance:** We will construct a fair division instance with \( n = 4r \) agents and \( m = 4r + s \) goods. The set of agents consists of \( 2r \) literal agents \( \{x_i, \bar{x}_i\}_{i \in [r]} \), and \( 2r \) dummy agents \( \{d_i, \bar{d}_i\}_{i \in [r]} \). The set of goods consists of \( 2r \) signature goods \( \{S_i, \bar{S}_i\}_{i \in [r]} \), \( s \) clause goods \( \{C_j\}_{j \in [s]} \), and \( 2r \) dummy goods \( \{T_i, B_i\}_{i \in [r]} \); here \( T_i \) and \( B_i \) denote the top and the bottom dummy goods associated with the variable \( X_i \), respectively.

**Preferences:** Table 1 shows the preferences of the agents. Let \( \succ \) define a reference ordering on the set of goods. For every \( i \in [r] \), if \( C_j \) and \( C_k \) denote the two clauses containing the positive literal \( x_i \), then the literal agent \( x_i \) ranks \( S_j \) at the top, and the clause goods \( C_j \) and \( C_k \) at ranks \( j + 1 \) and \( k + 1 \), respectively. The missing positions consist of remaining goods ranked according to \( \succ \) (we write \( \succ_{\ell} \) to denote the top \( \ell \) goods in \( \succ \) that have not been ranked so far). The symbol
The intractability in Theorem 3 persists even when we relax the fairness requirement from EFX to EFk.

**Theorem 4.** For any fixed $k \in \mathbb{N}$, determining the existence of an EFk and rank-maximal allocation is NP-complete.

We note that the proof of Theorem 4 (see the full version (Hosseini et al. 2020) for details) differs considerably from that of Theorem 3 as neither result is an immediate consequence of the other. Indeed, a YES instance of EFk+RM is also a YES instance of EFk+RM, but the same is not true for a NO instance.

A corollary of Theorem 4 is that checking the existence of EF1+RM allocations for additive valuations is also NP-complete. For this setting, Aziz et al. (2019) have shown NP-completeness even for three agents. By contrast, we will show that for lexicographic preferences, the problem is efficiently solvable when $n = 3$ (Proposition 5). The proof of this result is deferred to the full version of the paper (Hosseini et al. 2020).

**Proposition 5.** There is a polynomial-time algorithm that, given as input a lexicographic instance with three agents, computes an EF1 and rank-maximal allocation, whenever one exists.

Another avenue for circumventing the intractability in Theorem 3 is provided by maximin share guarantee (MMS). For additive valuations, EFX and MMS are incomparable notions (Amanatidis, Birmpas, and Markakis 2018). However, for lexicographic preferences, MMS is strictly weaker than EFX (see Proposition 6 in the full version (Hosseini et al. 2020) for details). This relaxation of EFX turns out to be computationally useful, as the existence of an MMS and RM allocations can be checked in polynomial time.

**Theorem 5.** There is a polynomial-time algorithm that, given as input a lexicographic instance, computes an MMS and rank-maximal allocation, whenever one exists.

We defer the proof of Theorem 5 to the full version (Hosseini et al. 2020), and briefly outline the algorithm below.

Fix any agent $i \in N$, and suppose its preference is given by $\succ_i := g_1 \succ g_2 \succ \ldots \succ g_m$. Under lexicographic preferences, the MMS partition of agent $i \in N$ is uniquely defined as $\{\{g_1\}, \{g_2\}, \ldots, \{g_{n-1}\}, \{g_n, g_{n+1}, \ldots, g_m\}\}$. This observation gives a characterization of MMS allocations: An allocation is MMS if and only if each agent either receives one or more of its top-$(n-1)$ goods, or it receives all of its bottom-$(m-n+1)$ goods.

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A table showing the preferences is included:

**Table 1: Preferences of agents in the proof of Theorem 3**

<table>
<thead>
<tr>
<th>$\succ$</th>
<th>$S_1 \succ S_1 \succ \ldots \succ S_r \succ S_r \succ T_1 \succ \ldots \succ T_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\succ C_1 \succ \ldots \succ C_s \succ B_1 \succ \ldots \succ B_p$</td>
<td></td>
</tr>
</tbody>
</table>

* indicates rest of the goods ordered according to $\succ$. The preferences of the (negative) literal agent $\pi_1$ and the dummy agents $d_i$, $\overline{d}_i$ are defined similarly as shown in Table 1. This completes the construction of the reduced instance.

Note that for any fixed $i \in [r]$, the signature good $S_i$ (or $\overline{S}_i$) is ranked at the top position by the literal agent $x_i$ (or $\pi_i$), and at a lower position by all other agents. Therefore, any rank-maximal allocation must assign $S_i$ to $x_i$ and $\overline{S}_i$ to $\pi_i$. For a similar reason, a rank-maximal allocation must assign the clause good $C_j$ to a literal agent corresponding to a literal contained in the clause $C_j$, and the dummy goods $T_i$, $B_i$ to the dummy agents $d_i$, $\overline{d}_i$. The aforementioned necessary conditions for rank-maximality are also sufficient since each clause good $C_j$ is ranked at the same position by all literal agents corresponding to the literals contained in clause $C_j$, and the goods $T_i$ and $B_i$ are ranked identically by $d_i$ and $\overline{d}_i$.

We will now argue the equivalence of solutions.

$(\Rightarrow)$ Given a satisfying truth assignment, the desired allocation, say $A$, can be constructed as follows: For every $i \in [r]$, assign the signature goods $S_i$ and $\overline{S}_i$ to the literal agents $x_i$ and $\pi_i$, respectively. If $x_i = 1$, then assign $T_i$ to $d_i$ and $B_i$ to $\overline{d}_i$, otherwise, if $x_i = 0$, then assign $T_i$ to $\overline{d}_i$ and $B_i$ to $d_i$. For every $j \in [s]$, the clause good $C_j$ is assigned to a literal agent $x_i$ (or $\pi_i$) if the literal $x_i$ (or $\pi_i$) is contained in the clause $C_j$ and the clause is satisfied by the literal, i.e., $x_i = 1$ (or $\pi_i = 1$). Note that under a satisfying assignment, each clause must have at least one such literal.

Observe that allocation $A$ satisfies the aforementioned sufficient condition for rank-maximality. Furthermore, any envied agent in $A$ receives exactly one good; in particular, if $d_i$ receives a bottom dummy good $B_i$, then we have $x_i = 0$ in which case the literal agent $x_i$, who is envied by $d_i$, does not receive any clause goods. By Proposition 1 $A$ is EFX.

$(\Leftarrow)$ Now suppose there exists an EFX and rank-maximal allocation $A$. Then, $A$ must satisfy the aforementioned necessary condition for rank-maximality. That is, for every $i \in [r]$, the signature goods $S_i$ and $\overline{S}_i$ are assigned to the literal agents $x_i$ and $\pi_i$, respectively (i.e., $S_i \in A_{x_i}$ and $\overline{S}_i \in A_{\pi_i}$), and the dummy goods $T_i$ and $B_i$ are allocated between the dummy agents $d_i$ and $\overline{d}_i$ (i.e., $\{T_i, B_i\} \subseteq A_{d_i} \cup A_{\overline{d}_i}$). In addition, for every $j \in [s]$, the clause good $C_j$ is assigned to a literal agent $x_i$ (or $\pi_i$) such that the literal $x_i$ (or $\pi_i$) is contained in the clause $C_j$. Also, by Proposition 1 each dummy agent must get exactly one dummy good.

We will construct a truth assignment for the (2/2/3)-SAT instance as follows: For every $i \in [r]$, if $T_i \in A_{d_i}$, then set $x_i = 1$, otherwise set $x_i = 0$. Note that the assignment is feasible as no literal is assigned conflicting values. To see why this is a satisfying assignment, consider any clause $C_j$. Suppose the clause good $C_j$ is assigned to a literal agent $x_i$ (an analogous argument works when $\pi_i$ gets $C_j$). Then, due to rank-maximality, we know that the literal $x_i$ must be contained in the clause $C_j$. Furthermore, since agent $x_i$ gets more than one good ($S_i, C_j \in A_{x_i}$), it cannot be envied under $A$ (Proposition 1). Thus, the dummy agent $d_i$ must get the top good $T_i$. Recall that in this case we set $x_i = 1$. Since clause $C_j$ contains $x_i$, it must be satisfied, as desired. 

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This is an additive valuations instance in which agent $i$ values its $j$th favorite good at $2^{m-n+1}$ equivalent to the lexicographic instance.
Construct a bipartite graph $G = (N \cup M, E)$ between agents and goods where an edge $(i, j) \in E$ exists if agent $i$ ranks good $j$ within its top-$(n-1)$ goods, and good $j$ can be ‘rank-maximally assigned’ to agent $i$. In other words, agent $i$ ranks good $j$ at least as high as any other agent.

If $G$ admits a perfect matching (this can be checked in polynomial time), then, by the above characterization, we have a partial allocation that is MMS and rank-maximal. By assigning the unmatched goods in a rank-maximal manner (which can be done in polynomial time), we obtain a desired complete allocation.

**Experiments**

We now revisit the non-existence result in Example 2 by examining how frequently $\{EF, EFX, EF1, MMS\} + \text{RM}$ allocations exist in synthetically generated data. To that end, we consider a fixed number of agents ($n = 5$) whose preferences over a set of $m$ goods (where $m \in \{5, \ldots, 15\}$) are generated using the Mallows model \cite{Mallows1957}. Given a reference ranking $\succ^* \in \mathcal{L}$ and a dispersion parameter $\phi \in [0, 1]$, the probability of generating a ranking $\succ_i \in \mathcal{L}$ under the Mallows model is given by $\frac{1}{Z} \phi^{d(\succ^*, \succ_i)}$, where $Z$ is a normalization constant and $d(\cdot)$ is the Kendall’s Tau distance. Thus, $\phi = 0$ induces identical preferences (i.e., $\succ_i = \succ^*$) while $\phi = 1$ is the uniform distribution. For each combination of $m$, $n$, and $\phi \in \{0, 0.25, 0.5, 0.75, 1\}$, we sample 1000 preference profiles, and use an integer linear program to check the existence of $\{EF, EFX, EF1, MMS\} + \text{RM}$ allocations. Code and data for all our experiments is available at https://github.com/sujoyksikdar/Envy-Freeness-With-Lexicographic-Preference.

Figure 2 presents our experimental results. For identical preferences ($\phi = 0$), every complete allocation is Pareto optimal as well as rank-maximal. Therefore, an EFX+RM (and hence $\{EF1, MMS\} + \text{RM}$) allocation always exists in this case, validating our theoretical result in Theorem 1. On the other hand, an EF+RM allocation fails to exist because of the conflict in top-ranked goods. At the other extreme for $\phi = 1$ (i.e., the uniform distribution), we note that the probability of existence of EF+RM outcomes grows steadily with $\phi$. This is because for (exact) envy-freeness, all five rankings should have distinct top goods, the probability of which is $(1 - \frac{1}{m}) \cdot (1 - \frac{1}{m}) \cdot (1 - \frac{1}{m}) \cdot (1 - \frac{1}{m}) \cdot (1 - \frac{1}{m})$. For $m = 100$, this value is roughly 0.9, suggesting that in the asymptotic regime, envy-free (and, by extension, EF+RM) allocations are increasingly likely to exist, and our algorithm in Proposition 2 will return EF+RM outcomes with high probability.

We observe that the gap between the fractions of instances that admit EFX+RM allocations and EF+RM allocations decreases with the number of goods. We conjecture that as the number of goods increases, the likelihood that every agent must be allocated more than one good in any RM allocation increases. Therefore, it is likely that envied agents receive more than one good, which is in direct conflict with EFX. In the full version of the paper \cite{Hosseini2020}, we test this conjecture through experiments for larger values of $m$, and observe an increasing trend in the fractions of instances that admit EF+RM allocations as well as those that admit EFX+RM allocations, while the gap between the two shrinks rapidly, and becomes negligible for larger number of goods. We also observe the general trend in our plots that $\{EF1, MMS\} + \text{RM}$ allocations tend to exist more frequently as the number of goods increases. Together, these observations suggest that the distributional approach could be a promising avenue for addressing the non-existence result in Example 2.

**Concluding Remarks**

We studied the interplay of fairness and efficiency under lexicographic preferences, obtaining strong algorithmic characterizations for EFX and Pareto optimality that addressed notable gaps in the additive valuations model, and outlining the computational limits of our approach for the stronger efficiency notion of rank-maximality. Going forward, it would be interesting to develop distribution-specific algorithms that can compute EFX+RM (or EF1+RM) allocations with, say, a constant probability. Extending our algorithmic characterization results to other fairness notions, specifically EF1 and MMS, will also be of interest.

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Supplementary Material for “Fair and Efficient Allocations under Lexicographic Preferences”

Proof of Proposition 3
Recall from Theorem 2 that the set of all mechanisms satisfying EFX, PO, strategyproofness, non-bossiness, and neutrality is characterized by the family of mechanisms in Algorithm 2 (parameterized by $\sigma$). In Proposition 3, we will show that the aforementioned set of properties is minimal in the sense that dropping any one of them necessarily expands the set of mechanisms beyond those in Algorithm 2.

Proposition 3. The set \{EFX, PO, strategyproofness, non-bossiness, neutrality\} is a minimal set of properties for characterizing the family of mechanisms in Algorithm 2.

Proof:
• EFX is necessary: Consider any serial dictatorship quota (SDQ) mechanism where the first agent has a quota of 2. From Proposition 2, we know that this mechanism is PO, strategyproof, non-bossy, and neutral. However, when agents have identical preferences, this mechanism fails to be EFX since the first agent, who is envied by everyone else, receives more than one good (Proposition 1).

• PO is necessary: A mechanism that leaves everything unassigned is EFX, strategyproof, non-bossy, and neutral, but clearly violates Pareto optimality.

• Non-bossiness is necessary: Consider the following "conditional" variant of Algorithm 2 with four agents and four goods: If the first two agents, $a_1$ and $a_2$, have identical preferences, then the ordering of agents is $\sigma = (a_1, a_2, a_3, a_4)$; otherwise, the priority ordering is $\sigma' = (a_1, a_2, a_4, a_3)$. It is easy to see that this mechanism is EFX (since each agent gets exactly one item), PO (because of sequencibility), and neutral (because each agent’s bundle is a singleton). It is also not hard to see that the mechanism is strategyproof: Indeed, agents $a_1$ and $a_2$ occur in the first and the second positions, respectively, under both orderings, and therefore cannot benefit by misreporting their preferences. Further, agents $a_3$ and $a_4$ cannot alter which ordering is invoked by misreporting, and for any fixed ordering, have no incentive to misreport either (since each instantiation of the mechanism is a serial dictatorship, which is strategyproof).

To see why this mechanism violates non-bossiness, consider the following preference profile:

$a_1: g_1 \succ g_2 \succ g_3 \succ g_4$
$a_2: g_1 \succ g_2 \succ g_3 \succ g_4$
$a_3: g_3 \succ g_4 \succ g_1 \succ g_2$
$a_4: g_3 \succ g_4 \succ g_1 \succ g_2$

Here, agent $a_2$ can change the allocation of agent $a_3$ by misreporting her preferences as $g_3 \succ g_1 \succ g_3 \succ g_4$ without changing her own allocation.

• Neutrality is necessary: Consider the following "conditional" variant of Algorithm 2 where agent 1 always picks a good first, and the ordering over the rest of the agents is decided as follows: if agent 1 picks $g_1$, then $\sigma = (1, 2, 3, \ldots, n)$. Otherwise, the ordering is $\sigma' = (1, n, n-1, \ldots, 2)$. It is easy to see that this mechanism is EFX, PO, strategyproof, and non-bossy, but not neutral.

• Strategyproofness is necessary: Consider the following mechanism variant of Algorithm 2: (i) Fix a priority ordering, say $\sigma = (1, \cdots, n)$, and execute one round of serial dictatorship according to $\sigma$. (ii) Assign the remaining goods according to the following rule: If agent $n$ envies agent $n-1$, all remaining goods are assigned to $n$. Otherwise, agents $n-1$ and $n$ pick the remaining goods in a round-robin fashion.

This mechanism is not strategyproof because in step (ii) the round-robin picking order is manipulable. In particular, agent $n$ can obtain all the goods that remain after one round of serial dictatorship by misreporting her preferences (and pretending to envy agent $n-1$).

It is easy to verify that the mechanism is EFX, PO, and neutral. We will now argue that this mechanism is also non-bossy. The intuitive idea is as follows: The first $n-2$ agents cannot change the outcome of any other agent by misreporting without changing their own allocation. Subject to the items allocated to these agents, the bundles of agents $n-1$ and $n$ are complements of each other, and thus, changing one implies changing the other. Formally, let us suppose, for the sake of contradiction, that the mechanism is bossy. There are two possible cases.

Case 1: Suppose agent $n$ envies $n-1$ under truthful reporting. (i) If $n-1$ misreports to avoid envy from agent $n$, then agent $n$ will pick the good that was assigned to agent $n-1$ under truthful reporting, and thus agent $n-1$’s outcome must change. (ii) If $n$ misreports so to avoid envy, then $n$’s outcome must change since it will now receive half of the remaining goods.

Case 2: Suppose agent $n$ does not envy agent $n-1$ under truthful reporting. (i) If $n$ reports his preferences to declare envy towards $n-1$, then his allocation must change since now he receives all the remaining goods, instead of receiving only half of the remaining goods under truthful reporting. (ii) Agent $n-1$ can only declare envy by picking a less desired good before agent $n$, and thus, changing its outcome, which contradicts bossiness.

Proof of Theorem 4
For any fixed $k \in \mathbb{N}$, the computational problem E$k$+RM-EXISTENCE asks whether a given instance admits an E$k$ and rank-maximal allocation (Definition 2). Note that the parameter $k$ is fixed in advance and is not a part of the input.

Definition 2 (E$k$+RM-EXISTENCE). Given any fair division instance with lexicographic preferences, does there exist an allocation that is envy-free up to $k$ goods (E$k$) and rank-maximal (RM)?

Theorem 4. For any fixed $k \in \mathbb{N}$, determining the existence of an E$k$ and rank-maximal allocation is NP-complete.

Proof. Given an allocation, one can check in polynomial time whether it satisfies E$k$ and rank-maximality. Hence, E$k$+RM-EXISTENCE is in NP.
To prove NP-hardness, we will show a reduction from \textsc{Partition into Triangles} (\textsc{PIT}), which is known to be \textsc{NP}-complete \parencite{Garey1979}. An instance of \textsc{PIT} consists of a graph $G = (V, E)$ with $|V| = 3q$ vertices for some $q \in \mathbb{N}$. The goal is to determine whether there exists a partition of the vertex set $V$ into $q$ disjoint sets $V^1, \ldots, V^q$ of three vertices each such that for every $V^i = \{v_{i1}, v_{i2}, v_{i3}\}$, the three edges $\{v_{i1}, v_{i2}\}$, $\{v_{i2}, v_{i3}\}$, and $\{v_{i3}, v_{i1}\}$ all belong to $E$.

We will assume that $G$ is a \textit{balanced tripartite} graph, that is, the set of vertices $V$ can be partitioned into three disjoint sets $W$, $X$, and $Y$ such that $|W| = |X| = |Y| = q$ and no pair of vertices within the same set are adjacent in $E$. It is known that \textsc{PIT} remains \textsc{NP}-complete even under this restriction \parencite{Custic2015} Proposition 5.1. We will write $W = \{w_1, \ldots, w_q\}$, $X = \{x_1, \ldots, x_q\}$, and $Y = \{y_1, \ldots, y_q\}$ to denote the vertices of the graph $G$. In addition, we will use $t := \binom{q}{2} - |E|$ to denote the number of \textit{non-edges} in $G$.

\textbf{Construction of the reduced instance:} We will construct a fair division instance with $n$ agents and $m$ goods, where $n = q + t(k + 2)q + 1$ and $m = 3q + (k - 1)q + t(k + 2)q + 1$.

The set of agents consists of $q$ \textit{main} agents $a_1, \ldots, a_q$, and $t$ groups of \textit{dummy} agents, each group comprising of $(k + 2)q + 1$ agents. The dummy agents in the $\ell$th group are denoted by $d_{\ell}^1, d_{\ell}^2, \ldots, d_{\ell}^{(k+2)q+1}$.

The set of goods consists of $3q$ \textit{main} goods $\{W_1, \ldots, W_q, X_1, \ldots, X_q, Y_1, \ldots, Y_q\}$, $(k - 1)q$ \textit{selector} goods and $t(k + 2)q + 1$ \textit{dummy} goods. The selector goods are classified into $q$ groups of $k - 1$ goods each, where the goods in the $\ell$th group are denoted by $S_{\ell}^1, S_{\ell}^2, \ldots, S_{\ell}^{k-1}$. The dummy goods are classified into $t$ groups of $(k + 2)q + 1$ goods each, where the goods in the $\ell$th group are denoted by $D_{\ell}^1, D_{\ell}^2, \ldots, D_{\ell}^{(k+2)q+1}$.

For any $i, j \in \mathbb{N}$ such that $i \leq j$, we will use the compact notation $D_{i:j}$ to denote the set $\{D_{i}^1, D_{i+1}^1, \ldots, D_{j}^1\}$. Similarly, we will write $S_{i:j}^\ell$, $W_{i:j}$, $X_{i:j}$, and $Y_{i:j}$ to denote $\{S_{i}^\ell, S_{i+1}^\ell, \ldots, S_{j}^\ell\}$, $\{W_i, W_{i+1}, \ldots, W_j\}$, $\{X_i, X_{i+1}, \ldots, X_j\}$, and $\{Y_i, Y_{i+1}, \ldots, Y_j\}$, respectively.

We will now describe the preferences of the agents. Let $\succ$ be a total ordering on the set of all goods given by

$$\succ : S_{1:k-1}^1 \succ S_{1:k-1}^2 \succ \ldots \succ S_{1:k-1}^{k-1} \succ W_{1:q} \succ X_{1:q} \succ Y_{1:q} \succ D_{1:(k+2)q+1}^1 \succ D_{2:(k+2)q+1}^1 \succ \ldots \succ D_{t:(k+2)q+1}^1,$$

where the goods within each group are ordered in $\succ$ according to their indices. Thus, for instance, the ordering within the group $S_{1:k-1}^1$ is $S_{1}^1 \succ S_{2}^1 \succ \ldots \succ S_{k-1}^1$. For every $i \in [q]$, the preferences of main agent $a_i$ are given by

$$a_i : S_{1:k-1}^i \succ S_{1:k-1}^{i+1} \succ \ldots \succ S_{1:k-1}^{k-1} \succ S_{1:k-1}^1 \succ S_{1:k-1}^2 \succ \ldots \succ S_{1:k-1}^{i-1} \succ W_{1:q} \succ X_{1:q} \succ Y_{1:q} \succ *,$$

where $*$ denotes the rest of the goods ordered according to $\succ$. Notice that the groups of selector goods are ordered in a cyclic manner by the main agents. In particular, the main agent $a_i$ prefers the selector group $S_{1:k-1}^i$ over any other set of goods.

In order to specify the preferences of the dummy agents, recall that there are as many groups of dummy agents as there are non-edges in $G$. Let the dummy agents in the $\ell$th group, namely $\{d_{\ell}^1, \ldots, d_{\ell}^{(k+2)q+1}\}$, be associated with the non-edge $(v_i, v_j) \notin E$, where $v_i, v_j \in W \cup X \cup Y$. All dummy agents in the $\ell$th group have identical preferences: For any $r \in \{1, \ldots, (k + 2)q + 1\}$, the preferences of the dummy agent $d_{\ell}^r$ are given by

$$\succ : D_{1:(k+2)q+1}^r \succ V_i \succ V_j \succ S_{1:k-1}^1 \succ \ldots \succ S_{1:k-1}^{q-1} \succ D_{(k+2)q+1}^r \succ *,$$

where $V_i$ and $V_j$ are the main goods associated with the vertices $v_i$ and $v_j$, respectively, and * once again denotes the rest of the goods ordered according to $\succ$. This completes the construction of the reduced instance.

Observe that all main goods are ranked inside the top $(k + 2)q$ positions by each main agent, whereas each dummy agent ranks any main good outside the top $(k + 2)q$ as a result, any rank-maximal allocation for the above instance must assign the main goods exclusively among the main agents. By similar reasoning, for every $i \in [q]$, the selector goods in the $\ell$th group, namely $S_{1:k-1}^i$, are assigned to the main agent $a_i$, and for every $\ell \in [t]$, all dummy goods in the $\ell$th group, namely $D_{1:(k+2)q+1}^\ell$, are assigned exclusively among the dummy agents in the $\ell$th group $\{d_{\ell}^1, \ldots, d_{\ell}^{(k+2)q+1}\}$. This necessary condition is also sufficient for rank-maximality, since the main agents have identical ranking of the main goods and the dummy agents in the $\ell$th group have identical ranking of the dummy goods in the $\ell$th group.

We will now argue the equivalence of the solutions.

$(\Rightarrow)$ Let $V^1 \sqcup V^2 \sqcup \ldots \sqcup V^q$ be a solution of \textsc{PIT} instance (where $\sqcup$ denotes disjoint union of sets). The desired allocation $A$ can be constructed as follows: For every $i \in [q]$, assign the main goods corresponding to the vertices in $V^i$ to the main agent $a_i$. Next, for every $i \in [q]$, assign the selector goods in the $\ell$th group, namely $S_{1:k-1}^i$, to the main agent $a_i$. Finally, for every $\ell \in [t]$ and every $i \in \{1, \ldots, (k + 2)q + 1\}$, assign the dummy good $D_{1}^\ell$ to the dummy agent $d_{\ell}^i$.

Note that the allocation $A$ assigns each good to exactly one agent; in particular, each main good is assigned to exactly one main agent since the sets $V^1, \ldots, V^q$ constitute a partition of the vertex set $V$. Additionally, $A$ satisfies the aforementioned sufficient condition for rank-maximality. All that remains to be shown is that $A$ is EF$k$.

Since each dummy agent $d_{\ell}^i$ gets exactly one good, the envy of any other agent towards $d_{\ell}^i$ can be eliminated by the removal of this good. Therefore, in order to establish that $A$ satisfies EF$k$, it suffices to bound the envy towards the main agents. Furthermore, each main agent $a_i$ receives its favorite good, namely $S_{1}^i$, under $A$, and therefore does not envy any other agent (because of lexicographic preferences). Thus, it suffices to focus only on the envy experienced by the dummy agents towards the main agents.
Fix a main agent \( a_r \). Suppose, for some \( \ell \in [t] \), some dummy agent in the \( \ell \)th group envies \( a_r \). Then, without loss of generality, the dummy agent \( d_{(k+2)q+1}^{\ell} \) must also envy \( a_r \). This is because all dummy agents in the same group have identical preferences and the \( d_{(k+2)q+1}^{\ell} \) receives the worst bundle among the agents in its group.

Recall that each group of dummy agents is associated with a non-edge. Let \( (v_i,v_j) \notin E \) be the non-edge associated with the \( \ell \)th dummy group. For a violation of EFk to occur between \( a_r \) and \( d_{(k+2)q+1}^{\ell} \), the agent \( a_r \) should receive \( k + 1 \) or more goods that are preferred by \( d_{(k+2)q+1}^{\ell} \) over its favorite good in its bundle, namely \( D_{(k+2)q+1}^{\ell} \). This is possible only if, in addition to \( k \) selector goods, \( a_r \) also receives the main goods \( V_i \) and \( V_j \) in the allocation \( A \).

Recall that the main goods assigned to the agent \( a_r \) come from the set \( V' \), which corresponds to a ‘triangle’ in the graph \( G \). Thus, each pair of main goods assigned to a main agent must correspond to vertices that constitute an edge in the graph \( G \). This, however, contradicts the fact that \( (v_i,v_j) \) is a non-edge. Therefore, the allocation \( A \) must satisfy EFk, as desired.

(\( \Leftarrow \)) Now suppose that there exists an allocation \( A \) that is EFk and rank-maximal. Then, \( A \) must satisfy the aforementioned necessary condition for rank-maximality. That is, all main goods are assigned among the main agents, the selector goods in the \( \ell \)th group are assigned to the main agent \( a_r \), and all dummy goods in the \( \ell \)th group are assigned among the dummy agents in the \( \ell \)th group.

We will now argue that each dummy agent must get at least one good in \( A \). Indeed, rank-maximality requires that each main agent is assigned \( k - 1 \) selector goods. In addition, since there are \( 3q \) main goods to be allocated among \( q \) main agents, some main agent, say \( a_r \), must get at least three main goods, resulting in a total of at least \( k + 2 \) goods in its bundle.

If some dummy agent, say \( d_{\ell}^{\ell} \), gets an empty bundle, then \( d_{\ell}^{\ell} \) will envy \( a_r \), and the envy cannot be eliminated by the removal of \( k \) goods, creating a violation of EFk. Thus, every dummy agent must get at least one good. Furthermore, for any \( \ell \in [t] \), there are \((k+2)q + 1\) dummy agents in the \( \ell \)th group and equally many dummy goods. Thus, each dummy agent gets exactly one dummy good in \( A \). Without loss of generality, we can assume that the agent \( d_{\ell}^{\ell} \) gets the good \( D_{\ell} \). In particular, the only good assigned to the agent \( d_{(k+2)q+1}^{\ell} \) is \( D_{(k+2)q+1}^{\ell} \).

We will now argue that each main agent gets exactly three main goods. By way of contradiction, suppose that some main agent, say \( a_r \), gets four or more main goods. Then, at least two of these goods, say \( V_i \) and \( V_j \), must both be \( W \)-goods (or \( X \)-goods or \( Y \)-goods). This means that the corresponding vertices \( v_i \) and \( v_j \) must both belong to either \( W \) or \( X \) or \( Y \), implying that \( (v_i,v_j) \) is a non-edge. By construction, each non-edge is associated with a group of dummy agents, so let the non-edge \( (v_i,v_j) \) be associated with the \( \ell \)th dummy group. This would imply that \( d_{(k+2)q+1}^{\ell} \) prefers \( k + 1 \) goods assigned to \( a_r \)—namely, the two main goods \( V_i \) and \( V_j \) as well as \( k - 1 \) selector goods \( S_{1:k-1}^{r} \)—over the only good in its own bundle, namely \( D_{(k+2)q+1}^{r} \). Therefore, the envy of \( d_{(k+2)q+1}^{r} \) towards \( a_r \) cannot be addressed by removing at most \( k \) goods from \( A_r \). This indicates a violation of EFk, which is a contradiction. Therefore, each main agent must get exactly three main goods. The above argument also establishes that these three main goods must come from three different sets \( W, X, \) and \( Y \). Thus, for every \( r \in [q] \), the bundle assigned to agent \( a_r \) is of the form \( A_r = \{ W_r, X_r, Y_r \} \).

We will now show that one can infer a solution of PIT from the allocation \( A \). Indeed, for every \( r \in [q] \), let \( V^r := \{ w_{h}, x_{i}, y_{j} \} \) whenever \( \{ W_{h}, X_{i}, Y_{j} \} \in A_r \). Notice that the sets \( V^1, \ldots, V^q \) constitute a valid partition of the vertex set of \( G \). This is because each main good is assigned to exactly one main agent in \( A \) and therefore each of the corresponding vertices is assigned to exactly one of the sets \( V^1, \ldots, V^q \). Furthermore, each set \( V^r = \{ v_{r,1}, v_{r,2}, v_{r,3} \} \) is a ‘triangle’, i.e., each pair of vertices in \( V^r \) must form an edge in \( G \), i.e., \( \{ v_{r,1}, v_{r,2}, v_{r,3} \}, \{ v_{r,1}, v_{r,3} \}, \{ v_{r,2}, v_{r,3} \} \in E \). Indeed, if some pair of vertices in \( V^r \) is a non-edge, then by earlier reasoning, some dummy agent will create an EFk violation with the main agent \( a_r \). Therefore, the sets \( V^1, \ldots, V^q \) constitute a valid solution of PIT.

This completes the proof of Theorem4 □

**Proof of Proposition5**

Recall the statement of Proposition5.

**Proposition 5.** There is a polynomial-time algorithm that, given as input a lexicographic instance with three agents, computes an EF1 and rank-maximal allocation, whenever one exists.

Proof. If the top-ranked goods of the three agents are all distinct, then an envy-free and rank-maximal allocation exists (Proposition4). Otherwise, suppose exactly two agents, say \( a_1 \) and \( a_2 \), have the same top-ranked good (the case where the top-ranked goods of all three agents coincide can be handled similarly).

We will start by carrying out all assignments that are uniquely determined by the rank-maximality condition. That is, if there is a unique agent for whom the highest ranking of a fixed good \( g \) is realized, then we assign \( g \) to that agent.

Next, let us consider two cases based on who out of \( a_1 \) or \( a_2 \) gets its top-ranked good, say \( g' \). If \( a_1 \) gets \( g' \), then \( a_2 \) is given the highest-ranked good in its list that it could be assigned without violating rank-maximality. If the corresponding partial allocation is EF1, then we can extend it to a rank-maximal allocation. Otherwise, we consider the other case where \( a_2 \) gets \( g' \) and \( a_1 \) gets the highest-ranked feasible good. If no EF1 partial allocation exists, the algorithm returns NO. □

**Comparing MMS with Relaxations of Envy-freeness**

Let us start by proving that EFX is a strictly stronger notion than MMS on the domain of lexicographic preferences.
**Proposition 6.** Given any instance with lexicographic preferences, any EFX allocation satisfies maxmin share guarantee (MMS) but the converse is not always true.

**Proof.** Suppose, for contradiction, that there is an EFX allocation $A$ that is not MMS. Then, there must exist some agent, say $i$, that receives a strict subset of its bottom-$(m-n+1)$ goods. (This is because under lexicographic preferences, if an agent receives one or more of its top-$(n-1)$ goods, or if it receives all of its bottom-$(m-n+1)$ goods, then its maxmin share guarantee is satisfied.)

Let $S$ denote the set of top-$(n-1)$ goods according to agent $i$’s preference. By the above observation, the goods in $S$ are allocated among the other $n-1$ agents. In order for the allocation $A$ to be EFX, no agent should get more than one good in $S$ (since we know from Proposition 1 that any envied agent gets exactly one good in an EFX allocation). Thus, each agent gets exactly one good in $S$, and therefore, agent $i$ envies every other agent.

Since agent $i$’s bundle is a strict subset of its bottom-$(m-n+1)$ goods, there must exist a good in $M \setminus S$ that is assigned to an agent other than $i$. This, however, results in an envied agent getting more than one good, thus violating the assumption that $A$ is EFX (Proposition 1). Therefore, $A$ must satisfy MMS.

To prove that MMS does not always imply EFX, consider the following instance with four agents and five goods:

$$
\begin{align*}
& a_1 : g_1 \succ g_2 \succ g_3 \succ g_4 \succ g_5 \\
& a_2 : g_1 \succ g_2 \succ g_3 \succ g_4 \succ g_5 \\
& a_3 : g_4 \succ g_1 \succ g_2 \succ g_3 \succ g_5 \\
& a_4 : g_5 \succ g_1 \succ g_2 \succ g_3 \succ g_4 \\
\end{align*}
$$

The allocation \{$\{g_1, g_2\}, \{g_3\}, \{g_4\}, \{g_5\}$ is MMS since each agent receives one or more of its top-$(n-1)$ goods. However, it is not EFX because agent $a_1$, who is envied by $a_2$, receives more than one good (Proposition 1).

It can also be shown that EF1 and MMS are incomparable notions in that one does not always imply the other. The fact that MMS does not imply EF1 follows from the example in the proof of Proposition 6. Indeed, agent $a_2$ continues to envy agent $a_1$ even after the removal of any good from the latter’s bundle. To prove that EF1 does not imply MMS, consider the following instance with identical preferences:

$$
\begin{align*}
& a_1 : g_1 \succ g_2 \succ g_3 \succ g_4 \succ g_5 \\
& a_2 : g_1 \succ g_2 \succ g_3 \succ g_4 \succ g_5 \\
& a_3 : g_4 \succ g_1 \succ g_2 \succ g_3 \succ g_5 \\
& a_4 : g_5 \succ g_1 \succ g_2 \succ g_3 \succ g_4 \\
\end{align*}
$$

The allocation \{$\{g_1\}, \{g_1, g_3\}, \{g_2\}, \{g_3\}$ is EF1; in particular, any agent’s envy towards $a_2$ can be eliminated by the removal of the good $g_1$. However, it fails MMS since $a_1$ gets a strict subset of its bottom-$(m-n+1)$ goods.

**MMS Characterization and Minimality**

When requiring PO, strategyproofness, non-bossiness, and neutrality, mechanisms satisfying MMS are restricted to those presented in Algorithm 2. This is due the fact that, subject to these additional properties, a mechanism satisfies EFX if and only if it satisfies MMS.

**Proposition 7.** An allocation of goods is MMS if and only if each agent’s bundle consists of one or more goods from among its top-$(n-1)$ goods or all of its bottom $m-n+1$ goods.

**Proof.** Consider the agent $i \in N$. Let $\succ_i = g_1 \succ g_2 \succ \cdots \succ g_m$. Then, the MMS partition of agent $i$ is uniquely defined, and its MMS value is given by MMS, $= \min \{ \{c_1\}, \{c_2\}, \ldots, \{c_n, \ldots, c_m\} \}$, where $\min\{}$ denotes the least-preferred bundle with respect to the lexicographic extension of $\succ_i$.

Therefore, we have the following characterization.

**Theorem 6.** For any ordering $\sigma$ of the agents, Algorithm 2 is MMS, PO, strategyproof, non-bossy, and neutral. Conversely, any mechanism satisfying these properties can be implemented by Algorithm 2 for some $\sigma$.

Notice that the EFX and MMS allocations do not necessarily coincide, but due to Proposition 2 under strategyproofness, non-bossiness, and neutrality, the set of mechanisms is restricted to SDQs, guaranteeing that EFX and MMS allocations do coincide. Nonetheless, although the family of mechanisms satisfying MMS or EFX along with PO, strategyproofness, non-bossiness, and neutrality are the same, the proof of minimality of these properties include subtle differences.

**Theorem 7.** The set of MMS, PO, strategyproofness, non-bossiness, and neutrality is a minimal set of properties for characterizing Algorithm 2.

**Proof.** The proofs of necessity of MMS, non-bossiness, and neutrality are identical to those in Proposition 5 by replacing EFX with MMS.

**PO is necessary.** Consider the following conditional mechanism: (i) Fix a priority ordering, say $\sigma = (1, \ldots, n)$, and execute one round of serial dictatorship according to $\sigma$. (ii) Assign the remaining goods according to the following rule: if agent $n$ receives a good that is ranked $n$ according to $\succ_n$, then all the remaining goods are assigned to $n$. Otherwise, throw away the remaining goods.

By Proposition 7 this mechanism is MMS. This rule is an SDQ mechanism with a quota of $(1, \ldots, m-n+1)$ or $(1, \ldots, 1)$. Thus, it is neutral, non-bossy, and strategyproof. However, the mechanism is not PO since it may throw away some goods if the quota of $(1, \ldots, 1)$ is selected.

**Strategyproofness is necessary.** Consider the following mechanism as a variant of Algorithm 2: (i) Fix a priority ordering, say $\sigma = (1, \ldots, n)$, and execute one round of serial dictatorship according to $\sigma$. (ii) Assign the remaining goods according to the following rule: if agent $n$ receives a good that is ranked $n$ according to $\succ_n$, then all the remaining goods are assigned to $n$. Otherwise, agents $n-1$ and $n$ pick the remaining goods in a round-robin fashion.

Notice that the above mechanism is MMS according to Proposition 7. The rest of the proof follows precisely as the one for Proposition 5.
Experiments

In Figure 2, we observe that the fraction of instances that admit an EFX+RM allocation follows a decreasing trend as the number of goods increase. We conjecture that this is because of the following reason: From Proposition 1, we know that under an EFX allocation, no envied agent can be assigned more than one good. Thus, in a regime where an EF allocation doesn’t exist, a sufficient condition for the non-existence of an EFX+RM allocation is that each agent is assigned two or more goods.

To test this conjecture experimentally, we find the fraction of “good” instances that admit an RM allocation where some agent receives at most one good, on the same data used in our experiments in Figure 2. The number of agents is fixed \( n = 5 \), and for each value of the dispersion parameter \( \phi \) of the Mallows model, we generate 1000 preference profiles over \( m \in \{5, \ldots, 50\} \) goods.

Figure 3 presents our experimental results: For each value of \( \phi \), a solid line represents the fraction of instances that are either “good” but do not admit an EF allocation or admit an EF+RM allocation, serving as an upper bound on the fraction of instances that admit an EFX+RM allocation. A dashed line represents the fraction of instances that admit an EF+RM allocation, and a dotted line represents the fraction of instances that do admit an EFX+RM allocation.

![Figure 3](image)

We observe that for \( \phi = 1 \), as the number of goods increases, the upper bound on the fraction of instances that admit EFX+RM allocations shows a decreasing trend. As a consequence, the shaded region representing the fraction of good instances that do not admit an EF allocation shrinks. We believe that this is the reason that as the number of goods increases, the fraction of instances that admit an EFX+RM allocation decreases even as the fraction of instances that admit an EF+RM allocation increases.