Binghamton University

EngiNet™

State University of New York

Thomas J. Watson

School of Engineering
and Applied Science

EngiNet™

WARNING

CS 460/560

Computer Graphics

Professor Richard Eckert

Lecture # 18

April 4, 2001

Lecture 18

- **Parametric Bezier Polynomial Curves**
- **Parametric B-Spline Polynomial Curves**

Bezier Polynomial Curves

- Parametric equations for a 2-D cubic polynomial curve:

  $x = ax*t^3 + bx*t^2 + cx*t + dx$

  $y = ay*t^3 + by*t^2 + cy*t + dy$

  $0<=t<=1$
- Shape of curve determined by polynomial coefficients:
  - (ax,bx,cx,dx, ay,by,cy,dy)

Control Points

- Want to easily determine shape of curve
- Specify four control points:

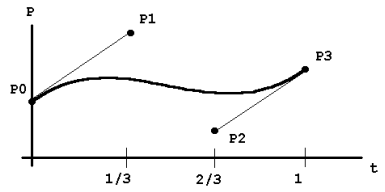
  P0 (x0,y0,z0), P1(x1,y1,z1), P2(x2,y2,z2), P3(x3,y3,z3)

Uniform Cubic Bezier Polynomial

$P = a*t^3 + b*t^2 + c*t + d, \quad 0 <= t <= 1$

- Control points uniformly separated in t

  P0 at t=0, P1 at t=1/3, P2 at t=2/3, P3 at t=1



Bezier Boundary Conditions

- Curve interpolates P0 at t=0
- Curve interpolates P3 at t=1
- Slope of curve at t=0 equals slope of line joining P0 and P1
- Slope of curve at t=1 equals slope of line joining P2 and P3

Apply BCs to Solve for Polynomial Coefficients

Final Result:

$$\begin{vmatrix} a \\ b \\ c \\ d \end{vmatrix} = \begin{vmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} * \begin{vmatrix} P0 \\ P1 \\ P2 \\ P3 \end{vmatrix}$$

$P = a*t^3 + b*t^2 + c*t + d, \quad 0 <= t <= 1$

Blending Function Representation

- Multiply matrix equation & rearrange
- Gives a different form:

$$P = \sum_{i=0}^{3} P_i * B_i(t)$$

- P_i: the control points (P0, P1, P2, P3)
- $B_i(t)$: "Bernstein Blending Functions"

- Blending Function form:
  - A weighted sum of the control points
  - Weighting factors: the Blending Functions
  - Value of Blending function gives "pull" of corresponding control point on curve at any point t
- The blending functions are given by:

  $B_i(t) = C(3,i) * t^i * (1-t)^{(3-i)}$
  - C(3,i) is the number of combinations of 3 things taken i at a time:
  - C(3,i) = 3! / (i! * (3-i)!)

Plotting Bezier Curve

1. Get control points P0=(x0,y0), P1=(x1,y1), P2=(x2,y2), P3=(x3,y3).
   - Could use interactive locator device (mouse)
2. Compute values of a, b, c, d from control points
   - Really ax,bx,cx,dx and ay,by,cy,dy
   - Use matrix equation and Forward Difference method

```
3. for (t=0; t<=1; t+=delta)
     Compute P (x & y) from polynomial equations
     if (t==0)
       MoveTo(x,y)
     else
       LineTo(x,y)
```

- Delta: a small increment (e.g. 0.05)
- Would give approximation to curve consisting of straight-line segments

Forward Differences

- Get new x,y values from old while stepping

  $x_{i+1} = x_i + \Delta x$
- Look at x equation:

  $x = at^3 + bt^2 + ct + d$
- Assume equal increments in t, $\delta t = \delta$

  $t_{i+1} = t_i + \delta, \quad \Delta x = x_{i+1} - x_i$

Final Results (recurrence relations):

$x = x + \Delta x$

$\Delta x = 3a\delta t^2 + (3a\delta^2 + 2b\delta)t + a\delta^3 + b\delta^2 + c\delta$

$\Delta x = \Delta x + \Delta^2 x$

$\Delta^2 x = 6a\delta^2 t + 6a\delta^3 + 2b\delta^2$

$\Delta^2 x = \Delta^2 x + \Delta^3 x$

$\Delta^3 x = 6a\delta^3$

Three adds on each iteration

Initial Values (t=t0)

- Need to calculate only once

$x0 = a*t0^3 + b*t0^2 + c*t0 + d$

$\Delta x0 = 3a\delta*t0^2 + (3a\delta^2 + 2b\delta)*t0 + a\delta^3 + b\delta^2 + c\delta$

$\Delta^2 x0 = 6a\delta^2*t0 + 6a\delta^3 + 2b\delta^2$

$\Delta^3 x0 = 6a\delta^3$

Higher Degree Bezier Curves

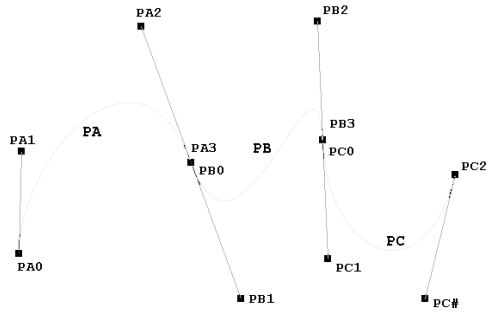
- Cubic (n=3) ==> 4 control points
- 4th degree ==> 5 control points
- nth degree ==> n+1 control points
- In general:

$$P(t) = \sum_{i=0}^{n} B^n_i(t) * P_i$$

$$B^n_i(t) = C(n,i) * t^i * (1-t)^{n-i}$$

- Higher Degree Bezier curves:
  - Can represent complex shapes
  - But moving any control point affects <u>entire</u> curve
  - Want <u>local control</u>
    - Moving a control point affects only one section of the curve
  - One way: use segmented Bezier curves

A Segmented Cubic Bezier Curve



Conditions at Knots

- Curves PA and PB
  - Determined by Control Points
    - PA0, PA1, PA2, PA3; PB0, PB1, PB2, PB3
- Level-0 continuity at knot:

  PA(at t=1) = PB(at t=0), i.e. at knot

  So PA3 = PB0 (Same control point)
- Level-1 continuity:

  dPA/dt(at t=1) = dPB/dt(at t=0)
  - So segments PA2-PA3 and PB0-PB1 must be colinear
  - (Recall Bezier Boundary Conditions)

B-Spline Polynomials

B-Spline Polynomials

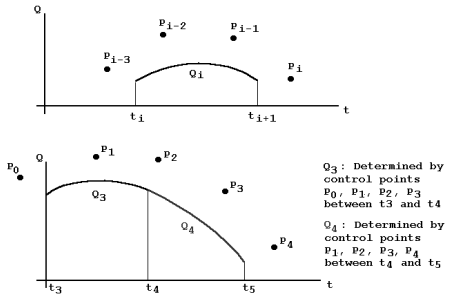
- Want local control
- Smoother curves
- B-spline curves:
  - Segmented approximating curve
  - 4 control points affect each segment
    - Local control
  - Level-2 continuity everywhere
    - Very smooth

Cubic B-Spline Polynomial Curves

- Approximate m+1 control points Pi (i=0,1,2,...m) with a curve consisting of m-2 cubic polynomial curve segments Q_i (i=3,4,...m)
- Each Q_i defined in terms of:
  - parameter t: $t_i <= t <= t_{i+1}$
  - and by four of the m+1 control points

- Segment Q_i determined by control points:
  P_{i-3}, P_{i-2}, P_{i-1}, P_i between t_i and t_{i+1}
- Q_i begins at $t = t_i$,
- Q_{i+1} joins Q_i at t_{i+1}
- Join point called a knot.
- For example
  - First segment is Q_3, begins at t_3, ends at t_4
  - Is determined by control points P0,P1,P2,P3
- Each segment is affected by only 4 control points
- Each control point affects only 4 curve segments

Uniform Cubic B-Spline Curves



Uniform Cubic B-Splines

- A special case where we assume that:
- $t_{i+1} = t_i + 1$
- Polynomial equation for segment Qi:
- $Qi(t) = a*(t-ti)^3 + b*(t-ti)^2 + c*(t-ti) + d$, $ti <= t <= ti+1$
- Take independent variable as t-ti
  - Will vary from 0 to 1 for any interval

- Need to get polynomial coefficients (a,b,c,d)
  - from control points
- Find a "B-Spline Basis Matrix"
  - as for Bezier curves
  - but must do computation for each interval

$$\begin{vmatrix} a \\ b \\ c \\ d \end{vmatrix} = M_{BS} * \begin{vmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{vmatrix}$$

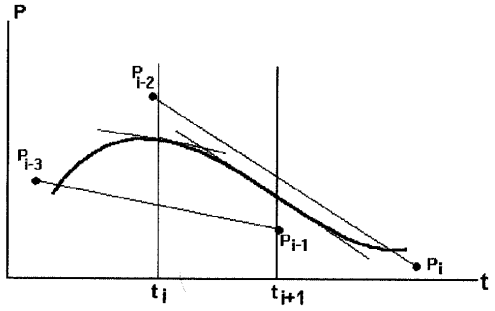
- M_{BS} is the desired matrix

B-Spline Continuity Conditions

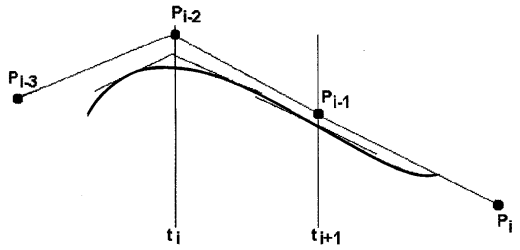
- Conditions on 1st & 2nd derivatives:

1. dQ_i/dt (at $t=t_i$) = slope of line segment joining P_{i-3} and P_{i-1}
2. dQ_i/dt (at $t=t_{i+1}$) = slope of line segment joining P_{i-2} and P_i
3. $(dQ_i/dt)'$ $(t=t_i)$ = change in slope at t_i= (slope of P_{i-3}-P_{i-2} - slope of P_{i-2}-P_i) / Δt
4. $(dQi/dt)'$ $(t=t_{i+1})$ = change in slope at t_{i+1} = (slope of P_{i-2}-P_{i-1} - slope of P_{i-1}-P_i) / Δt

Continuity Conditions 1 and 2



Continuity Conditions 3 and 4



$dQi/dt = 3*a*(t-ti)^2 + 2*b*(t-ti) + c$

$(dQi/dt)' = 6a*(t-ti) + 2*b$

- Condition 1: $c = (P_{i-1} - P_{i-3})/2$
- Condition 2: $3*a + 2*b + c = (P_i - P_{i-2})/2$
- Condition 3: $2*b = ((P_{i-1} - P_{i-2}) - (P_{i-2} - P_{i-3})) / 1$
- Condition 4: $6*a + 2*b = ((P_i-P_{i-1})-(P_{i-1}-P_{i-2})) / 1$
- These four equations are not independent
  - Solving gives only a, b, c, but not d

Solution

$a = (1/6) * (-P_{i-3} + 3*P_{i-2} - 3*P_{i-1} + P_i)$

$b = (1/6) * (3*P_{i-3} - 6*P_{i-2} + 3*P_{i-1})$

$c = (1/6) * (-3*P_{i-3} + 3*P_{i-1})$

Need another condition to get d

- Choose the following condition:

  Q (at $t=ti$) $= (1/6) * (P_{i-3} + 4*P_{i-2} + P_{i-1})$
  - i.e., control point at ti (Pi-2) pulls 4 times as hard at t=ti as control points on either side of ti
- Substitute in polynomial equation -->

  $d = (1/6) * (P_{i-3} + 4*P_{i-2} + P_{i-1})$

Uniform Cubic B-Spline Coefficient Matrix Equation

$$\begin{vmatrix} a \\ b \\ c \\ d \end{vmatrix} = (1/6) * \begin{vmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{vmatrix} * \begin{vmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{vmatrix}$$

Could also be written in terms of blending functions

$$Q_i(t) = \sum_{j=0}^{3} B_{i-j,4}(t) * P_{i-j}$$

Bi-3,4 = 1/6 * (1-t)^3

Bi-2,4 = 1/6 * (3t^3 - 6t^2 + 4)

Bi-1,4 = 1/6 * (-3t^3 + 3t^2 -3T +1)

Bi,4 = 1/6 * t^3

See Foley & Van Dam

Plotting Uniform Cubic B-Splines

- Given m+1 control points P0,P1,P2,...Pm
  - (Recall that each has an x and y coordinate
    - i.e., P0 --> x0 and y0, etc.)
- The following is a "brute force" algorithm to plot the curve
  - delta is a very small increment (e.g., 0.05)

Brute Force Algorithm

```
For (i=3 to m)
    Compute ax,bx,cx,dx and ay,by,cy,dy
        from control points i-3, i-2, i-1, i
    For (t=0; t<=1; t+=delta)
      x = ax*t^3 + bx*t^2 + cx*t + dx
      y = ay*t^3 + by*t^2 + cy*t + dy
      If (t==0)
        MoveTo(x,y)
      Else
        LineTo(x,y)
```

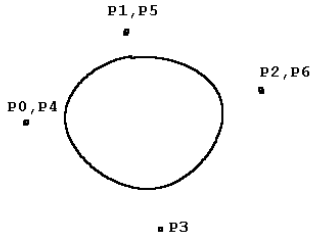
- To increase performance, use forward differences

Closed Cubic B-Splines

Make last 3 control points coincide with 1st 3

0 <--> m-2, 1 <--> m-1, 2 <--> m

Example: m=6



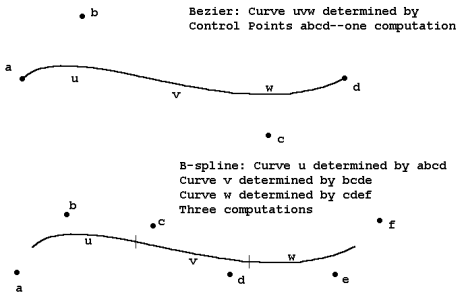
Forcing Interpolation

Reproduce a control point three times

Properties of Uniform B-Splines

1. Local Control
   Each segment determined by only 4 control points
2. Approximates control points; doesn't interpolate
   (However it will interpolate triplicated control points)
3. Lies inside convex hull of control points
4. Invariant under affine transformations
5. Very smooth
   Level-2 continuity everywhere
6. More computations required than for "equivalent" Bezier curve

Bezier vs. B-Spline Curves



Non-uniform Cubic B-Splines

- Greater variety of curve shapes
- Can have cusps and other discontinuities
- Intervals between successive knots varies
- Knot values must be specified

$t_0, t_1, t_2, t_3, t_4, \ldots t_{m+4}$

NON-UNIFORM CUBIC B-SPLINES

Variable size intervals between successive knot values

Must specify knot values --> the knot vector,
a non-decreasing sequence
e.g., (0,0,0,0,1,1,2,3,4,4,....)

Can have multiple knots

The curve segment Q_i is determined by control points: P_{i-3}, P_{i-2}, P_{i-1}, P_i

and by blending functions: $B_{i-3,4}(t)$, $B_{i-2,4}(t)$, $B_{i-1,4}(t)$, $B_{i,4}(t)$

[4 = the order (degree-3 plus 1) of the polynomials]

is given by:

$$Q_i(t) = P_{i-3}\,B_{i-3,4}(t) + P_{i-2}\,B_{i-2,4}(t) + P_{i-1}\,B_{i-1,4}(t) + P_i\,B_{i,4}(t)$$

$3 \le i \le m$, $t_i \le t < t_{i+1}$ defined between t_3 and t_{m+1}

If $t_i = t_{i+1}$ then the curve segment Q_i degenerates to a point.

The Blending functions B(t) are defined recursively:

$$B_{i,1}(t) = \begin{cases} 1, & t_i \le t < t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

$$B_{i,2}(t) = \frac{t - t_i}{t_{i+1} - t_i} B_{i,1}(t) + \frac{t_{i+2} - t}{t_{i+2} - t_{i+1}} B_{i+1,1}(t)$$

$$B_{i,3}(t) = \frac{t - t_i}{t_{i+2} - t_i} B_{i,2}(t) + \frac{t_{i+3} - t}{t_{i+3} - t_{i+1}} B_{i+1,2}(t)$$

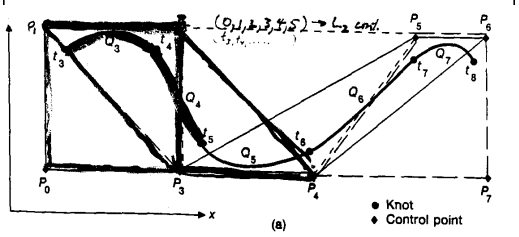
$$B_{i,4}(t) = \frac{t - t_i}{t_{i+3} - t_i} B_{i,3}(t) + \frac{t_{i+4} - t}{t_{i+4} - t_{i+1}} B_{i+1,3}(t)$$

In these equations, 0/0 is defined to be equal to 0.

Case A (Level-2 Continuity)

- Knot vector: (0,1,2,3,4,5,...)
  - Just our friend the uniform B-spline
- Q3 determined by P0, P1, P2, P3
- Q4 determined by P1, P2, P3, P4
- Q3 and Q4 share control points P1,P2,P3
  - Three constraints ==> L0, L1, L2 continuity

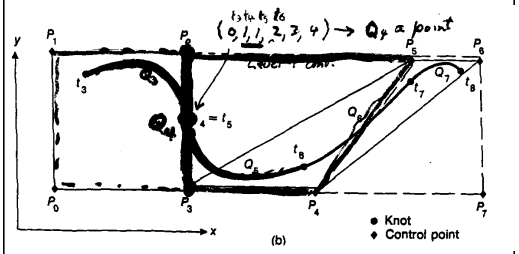
Case A (Level-2 Continuity)



Case B (Level-1 Continuity)

- Knot vector: (0,1,1,2,3,4,...)
- Segment Q4 becomes a point
  - (since t4 = t5)
- Q3 determined by P0, P1, P2, P3
- Q5 determined by P2, P3, P4, P5
- So Q4 must lie on line connecting P2 & P3
- Q3 and Q5 share control points P2 & P3
  - Two constraints ==> L0, L1 continuity

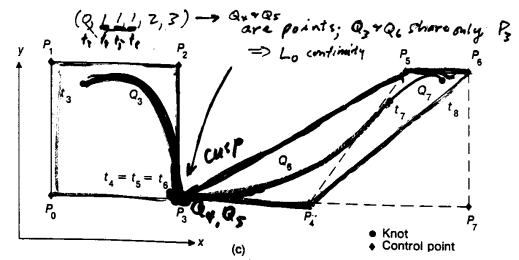
Case B (Level-1 Continuity)



Case C (Level-0 continuity)

- Knot vector: (0,1,1,1,2,3,...)
- Q4 and Q5 become points
  - (since t4=t5=t6)
- Q3 determined by P0, P1, P2, P3
- Q6 determined by P3, P4, P5, P6
- So Q4/Q5 must lie on control Point P3
  - (interpolates it)
- Q3 and Q6 share control point P3
  - One constraint ==> L0 continuity

Case C (Level-0 continuity)



Case D (No Continuity)

- Knot vector: (0,1,1,1,1,2,...)
- Q4, Q5, Q6 become points
  - (since t4=t5=t6=t7)
- Q3 determined by P0, P1, P2, P3
- Q7 determined by P4, P5, P6, P7
- There is no overlap
- Q3 and Q7 share no control points
  - No constraints ==> discontinuity

Case D (No Continuity)