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WARNING

CS 460/560

Computer Graphics

Professor Richard Eckert

Lecture # 17

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Lecture 16

- **Parametric Bezier Polynomial Curves**
- **Parametric B-Spline Polynomial Curves**

Parametric Curves

- x and y expressed as explicit functions of a parameter, t
  - x = f(t)
  - y = g(t)
- Range of parameter also given
  - Delimits the extent of the curve
- To plot, let t vary over its range
  - Points on curve are generated
- Easily extended to curves in 3-D
  - z = h(t)

Polynomials

- Explicit Form of n-degree polynomial:
  - $y = a_0 + a_1*x + a_2*x^2 + ... a_n*x^n$
- Assume we have a set of n+1 known control points: (xi,yi)
- Get polynomial coefficients a_i from the control points
- Two Methods:
  - Interpolation
  - Approximation

Approximating Polynomials

- Curve determined by control point
- But does NOT go through all of them
- Control Points act as magnets
- Better for many graphics applications
- Most commonly used:
  - Bezier curves
  - B-spline curves

- Bezier Curves
  - See CS-460/560 Notes:
  - Bezier Polynomial Curves

  http://www.cs.binghamton.edu/~reckert/460/bezier.htm
- B-Spline Curves
  - See CS-460/560 Notes:
  - B-spline Polynomial Curves

  http://www.cs.binghamton.edu/~reckert/460/bspline.htm

Bezier Polynomial Curves

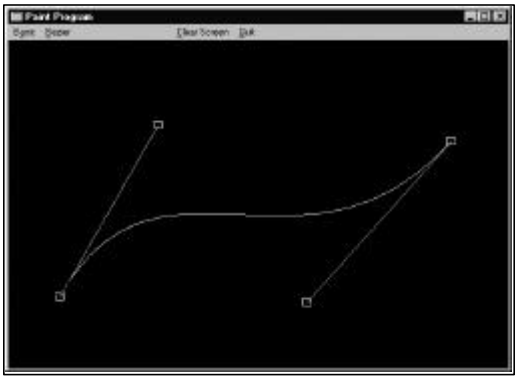
- Parametric equations for a 2-D cubic polynomial curve:
  - $x = ax*t^3 + bx*t^2 + cx*t + dx$
  - $y = ay*t^3 + by*t^2 + cy*t + dy$
  - $0<=t<=1$
- Shape of curve determined by polynomial coefficients:
  - (ax,bx,cx,dx, ay,by,cy,dy)

Easily extended to 3-D

- Just add a third parametric equation:
  $z = az*t^3 + bz*t^2 + cz*t + dz$

Control Points

- Want to easily determine shape of curve
- Specify four control points:
  P0 (x0,y0,z0), P1(x1,y1,z1), P2(x2,y2,z2), P3(x3,y3,z3)
- Could use interpolating polynomial
- More useful: approximating polynomial
  - Doesn't interpolate all control points
  - Many ways to do the approximating



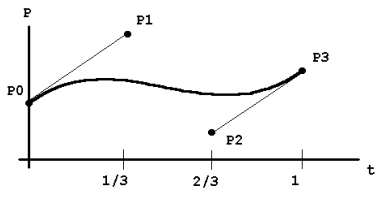
Uniform Cubic Bezier Polynomial

- Important kind of approximating polynomial
- Assume a generic parametric cubic polynomial:
  $P = a*t^3 + b*t^2 + c*t + d, \quad 0 <= t <= 1$
- Determined by control points P0, P1, P2, P3
  - P could be x, y, or z
  - a could be ax, ay, or az
  - P0 could be x0, y0, z0, etc.

Uniform Bezier Polynomial

$P = a*t^3 + b*t^2 + c*t + d, \quad 0 <= t <= 1$

- Control points uniformly separated in t
  P0 at t=0, P1 at t=1/3, P2 at t=2/3, P3 at t=1

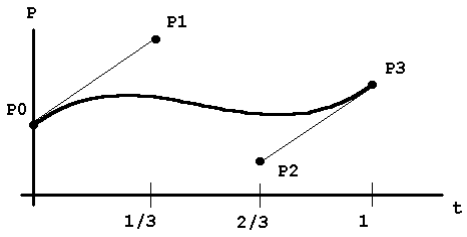


Boundary conditions:

$P = a*t^3 + b*t^2 + c*t + d, \quad 0 <= t <= 1$

1. Curve must interpolate control point P0
   P=P0 when t=0
   So P0 = d
2. Curve must interpolate control point P3
   P=P3 when t=1
   so P3 = a + b + c + d

Uniform Cubic Bezier Curve



$P = a*t^3 + b*t^2 + c*t + d, \quad 0 <= t <= 1$

3. Slope of curve at t=0 must be equal to that of the line that joins control points P0 and P1
   - dP/dt(at t=0) = slope of P0-P1
   - $dP/dt = 3*a*t^2 + 2*b*t + c$
   - slope of P0-P1 = (P1-P0)/(1/3-0)
   - So: c = 3*(P1-P0)
4. Slope of curve at t=1 must be equal to that of the line that joins control points P2 and P3
   - dP/dt(at t=1) = slope of P2-P3
   - 3*a + 2*b + c = (P3-P2)/(1 - 2/3)
   - 3*a + 2*b + c = 3*(P3-P2)

Solving for Polynomial Coefficients

- Equations:

```
0   +  0   +  0   +   d   = P0
a   +  b   +  c   +   d   = P3
0   +  0   +  c   +   0   = 3*(P1-P0)
3*a +  2*b +  c   +   0   = 3*(P3-P2)
```

This can be expressed in matrix form:

```
| 0  0  0  1 |   | a |   | P0        |
| 1  1  1  1 | * | b | = | P3        |
| 0  0  1  0 |   | c |   | 3*(P1-P0) |
| 3  2  1  0 |   | d |   | 3*(P3-P2) |
```

In other words:

- A * C = V
- C = [a, b, c, d], the coefficient vector
- V = [P0, P3, 3*(P1-P0), 3*(P3-P2)]
- A = the above 4X4 matrix

- To solve multiply by A-inverse
  - $A^{-1} * A * C = A^{-1} * V$
  - $C = A^{-1} * V$
- Use Gauss-Jordan elimination or other techniques to get A-inverse
- Result:

$$A^{-1} = \begin{vmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

So: $C = A^{-1} * V$

$$C = \begin{vmatrix} a \\ b \\ c \\ d \end{vmatrix} = \begin{vmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} * \begin{vmatrix} P0 \\ P3 \\ 3(P1-P0) \\ 3(P3-P2) \end{vmatrix}$$

Final Result:

$$\begin{vmatrix} a \\ b \\ c \\ d \end{vmatrix} = \begin{vmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} * \begin{vmatrix} P0 \\ P1 \\ P2 \\ P3 \end{vmatrix}$$

Uniform Bezier Result

- A constant 4 X 4 matrix multiplied by a vector whose components are the control points
- Constant matrix called the Bezier geometry matrix
- Other kinds of polynomial curves will have their polynomial coefficients given by a similar equation
  - Matrix elements of the constant 4 X 4 geometry matrix will change

Writing Bezier Result in Compact Form

- Points P on curve are given by:
  $P = a*t^3 + b*t^2 + c*t + d, \quad 0 <= t <= 1$
- Can be written in a more compact form:
  $P = T * Bg * Pc$
  T: row vector of parameter powers $|t^3 \ t^2 \ t \ 1|$
  Bg: the constant 4 X 4 Bezier Geometry matrix
  Pc: column vector of the control points

Blending Function Representation

- Multiply matrix equation & rearrange
- Gives a different form:

$$P = \sum_{i=0}^{3} P_i * B_i(t)$$

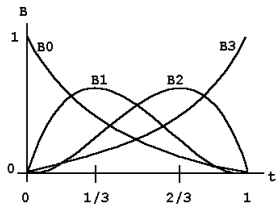
- P_i: the control points (P0, P1, P2, P3)
- $B_i(t)$: "Bernstein Blending Functions"

- Blending Function form:
  - A weighted sum of the control points
  - Weighting factors: the Blending Functions
  - Value of Blending function gives "pull" of corresponding control point on curve at any point t
- The blending functions are given by:
  $B_i(t) = C(3,i) * t^i * (1-t)^{(3-i)}$
  - C(3,i) is the number of combinations of 3 things taken i at a time:
  - $C(3,i) = 3! / (i! * (3-i)!)$

The Berstein Blending Functions

- For the cubic Bezier polynomial:
  $B_0(t) = (1-t)^3$
  $B_1(t) = 3 * t * (1-t)^2$
  $B_2(t) = 3 * t^2 * (1-t)$
  $B_3(t) = t^3$

The Berstein Blending Functions



$B0 = (1-t)^3$
t=0 --> 1, t=1 --> 0

$B1 = 3*t*(1-t)^2$
Maximum at t=1/3

$B2 = 3*t^2*(1-t)$
Maximum at t=2/3

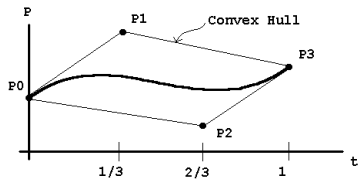
$B3 = t^3$
t=0 --> 0, t=1 --> 1

- B0 has maximum value of 1 (100%) at t=0
  - All other blending functions give 0 there
    - Control pt. P0 pulls with 100% "force" at t=0
    - None of the other control points pulls at all
    - So curve must go through P0 (as we know)
- B3 has maximum value of 1 (100%) at t=1
  - All other blending functions give 0 there
  - So curve must go through P3

- B1 has its maximum value at t=1/3
  - Value is less than 1
  - Other Blending functions non-zero but with values < B1's
  - So curve cannot pass through P1
  - Curve pulled hardest by P1
- Similarly at t=2/3, P2 pulls hardest

Properties of Bezier Curves

- Bk <=1, so:
  - Control points lie outside curve
    - curve lies inside "Convex Hull" of control points
  - Important for clipping



More Bezier Curve Properties

- "Pull" of a control point is proportional to "distance" (in t) from the control point
- Bezier Curves are invariant under affine transformations
  - So just transform control points and redraw the curve

Plotting Bezier Curves

- Brute Force Method:
  1. Get control points P0=(x0,y0), P1=(x1,y1), P2=(x2,y2), P3=(x3,y3).
     - Could use interactive locator device (mouse)
  2. Compute values of a, b, c, d from control points
     - Really ax,bx,cx,dx and ay,by,cy,dy
     - Use matrix equations
     - (Alternative: use blending functions)

```
3. for (t=0; t<=1; t+=delta)
     Compute P (x & y) from polynomial equations
     if (t==0)
       MoveTo(x,y)
     else
       LineTo(x,y)
```

- Delta: a small increment (e.g. 0.05)
- Would give approximation to curve consisting of straight-line segments

Improving Performance

- Brute force is much too much work
  $P = a*t^3 + b*t^2 + c*t + d$
  - Each iteration: 5 floating point multiplies
    c*t, t*t, b*(t*t), t*(t*t), a*(t*(t*t))]
  - and 3 floating point adds
- Using Horner's rule for polynomial evaluation:
  $P = ((a*t+b)*t+c)*t$
  - 3 multiplies and 3 adds

- Can do much better
- Use technique of Forward Differences
- Will improve performance
  - only 3 floating point adds during each iteration!

Forward Differences

- Get new x,y values from old while stepping
  $x_{i+1} = x_i + \Delta x$
- Look at x equation:
  $x = at^3 + bt^2 + ct + d$
- Assume equal increments in t, $\delta t=\delta$
  $t_{i+1} = t_i + \delta, \quad \Delta x = x_{i+1} - x_i$
  $\Delta x = a(t+\delta)^3 + b(t+\delta)^2 + c(t+\delta)t + d - at^3 - bt^2 - ct - d$
- Result (first forward difference):
  $\Delta x = 3a\delta t^2 + (3a\delta^2 + 2b\delta)t + a\delta^3 + b\delta^2 + c\delta$
  Reduced to quadratic

- Do again to simplify Δx
  $\Delta x = \Delta x + \Delta(\Delta x) = \Delta x + \Delta^2 x$
  $\Delta^2 x = \Delta x(t+\delta) - \Delta x(t)$
  $\Delta^2 x = 3a\delta(t+\delta)^2 + (3a\delta^2+2b\delta)(t+\delta) + k$
  $-3a\delta t^2 - (3a\delta^2 + 2b\delta)t - k$
  where $k = a\delta^3 + b\delta^2 + c\delta$
- Result (second forward difference):
  $\Delta^2 x = 6a\delta^2 t + 6a\delta^3 + 2b\delta^2$
  Reduced to linear equation
  For next step let $k1 = 6a\delta^3 + 2b\delta^2$

- Do again to simplify $\Delta^2 x$
  $\Delta^2 x = \Delta^2 x + \Delta(\Delta^2 x) = \Delta^2 x + \Delta^3 x$
  $\Delta^3 x = \Delta^2 x(t+\delta) - \Delta^2 x(t)$
  $\Delta^3 x = 6a\delta^2(t+\delta) + k1 - 6a\delta^2 t - k1$
- Result (third forward difference):
  $\Delta^3 x = 6a\delta^3$
- Final Results(recurrence relations):
  $x = x + \Delta x$
  $\Delta x = \Delta x + \Delta^2 x$
  $\Delta^2 x = \Delta^2 x + \Delta^3 x$
  Three adds on each iteration

Initial Values

- Need to calculate only once

$x0 = a*t0^3 + b*t0^2 + c*t0 + d$

$\Delta x0 = 3a\delta*t0^2 + (3a\delta^2 + 2b\delta)*t0 + a\delta^3 + b\delta^2 + c\delta$

$\Delta^2 x0 = 6a\delta^{2*}t0 + 6a\delta^3 + 2b\delta^2$

$\Delta^3 x0 = 6a\delta^3$

Example Using Forward Difference Calculations

$x = t^3 + 2t^2 + 3t + 1,\ 0<=t<=10$

For simplicity take $\delta = 1$

a=1, b=2, c=3, d=1

t0=0

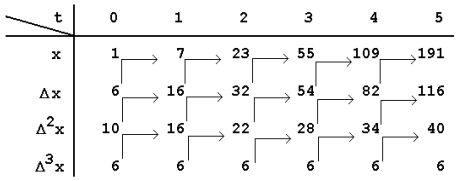
x0 = 1

$\Delta x0 = 1 + 2 + 3 = 6$

$\Delta^2 x0 = 6 + 2*2 = 10$

$\Delta^3 x0 = 6$

Example Forward Difference Calculations



t	0	1	2	3	4	5
x	1	7	23	55	109	191
Δx	6	16	32	54	82	116
$\Delta^2 x$	10	16	22	28	34	40
$\Delta^3 x$	6	6	6	6	6	6