Fractals

Fractals

- Beautiful designs of infinite structure and complexity
- Qualities of Fractals:
  - Fractional dimension
  - Self similarity
  - Complex structure at all scales
  - Chaotic dynamical behavior
  - Simple generation algorithms
  - Capable of describing an enormous range of natural objects

Some Objects Representable by Fractals

- Mountains
- Clouds
- Snow flakes
- Fog
- Frost patterns
- Fire
- River basins
- Sea coasts

- Explosions and fireworks
- Plants
- Island formations
- Galaxies
- Arteries and veins
- Cells
- Rivers
- Stock market fluctuations
- Weather systems
- Many More!!

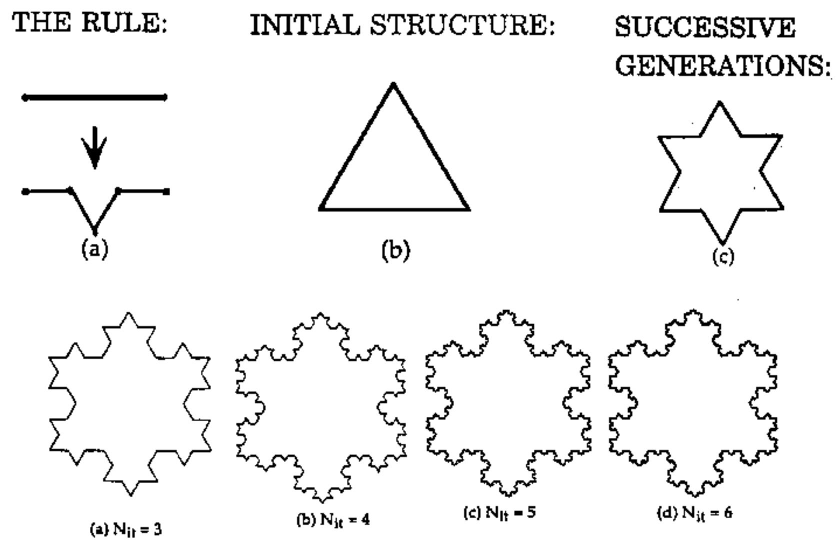
Types of Fractal-Generation Algorithms

- Linear Replacement Mapping
- Iterated Function Systems
- Random Midpoint Displacement
- Plasmas
- Escape-time algorithms
- Complex plane mapping
- Recursive, grammar-based systems
- Particle Systems

Linear Replacement Mapping

1. Define initial structure in terms of line endpoints
2. Define a replacement mapping
   - rule that replaces each line with a refined set of lines
   - defines next generation of structure
   - inherently recursive
3. Iterate the refinement until desired level achieved

Example: Koch Snowflake

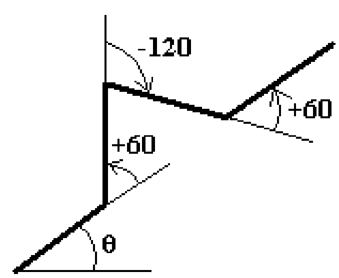


Implementing a Koch Curve

Assume recursive function Koch(len,theta,n)
(len = length, theta = angle of line, n=recursion level)

To get next generation curve (i.e., if n >0) from a line segment, make 4 calls:

```
Koch (len/3, theta, n-1);
theta += 60;
Koch (len/3, theta, n-1);
theta -= 120;
Koch (len/3, theta, n-1);
theta += 60;
Koch (len/3, theta, n-1);
```



Base case: At lowest (n=0) level of recursion, so draw line:

```
LineTo (len*cos(theta), len*sin(theta) ) ;
```

Using the Koch() function

- 1. Assign a value to n and an initial position (x0,y0)
- 2. Make a call to MoveTo(x0, y0)
- 3. Assign an initial len, and theta
- 4. Make the call Koch (len, theta, n)

FractInt

- Classic free program for playing around with many different kinds of fractals
- Originally a DOS program
- Has been extended to Windows
- FractInt home page:
  - http://spanky.triumf.ca/www/fractint/fractint.html
  - Has a link to a download site

Dimension of a Fractal

- Look at a non-fractal, a line (1-D)
  - Subdivide into N similar pieces, e.g., 3
  - Reduce by a scaling factor r, e.g., 1/3

    $1 = N*r^1$
- Another: a square (2-D): r = 1/3, N = 9

    $1 = N*r^2$
- Another: a cube (3-D): r = 1/3, N = 27

    $1 = N*r^3$
- Evidently the exponent of r is the "dimension" of the object

Hausdorff Dimension

- In general, assume $1 = N*r^D$
  - where D is the "dimension" of the object
- Solve for D:
  - $D = \log(N)/\log(1/r)$
- For a Koch curve
  - N=4, r=1/3
  - D = log(4)/log(3) = 1.2857
  - Non-integer!!
- Somehow it occupies more space than a linear object in Euclidean space
- Fractals: Hausdorff dim. > topological dim.

Iterated Function Systems

- Define a set of contractive affine transformation matrices Mi:

$$M_i = \begin{bmatrix} a_i & b_i & e_i \\ c_i & d_i & f_i \\ 0 & 0 & 1 \end{bmatrix}$$

Generate new points P'=(x',y') from old P=(x,y):

P' = Mi*P

i.e.:

x' = ai*x + bi*y + ei

y' = ci*x + di*y + fi

The IFS Algorithm

Select “seed point” (x,y)

Repeat many time:

Pick an i randomly

Compute x',y' from x,y using Mi (ai,bi,ci,di,ei,fi)

Plot (x',y') on screen

Set (x,y) to (x',y')

Accelerating the IFS Algorithm

- Choose each Mi with a probability:

$$P_i = \frac{| a_i d_i - b_i c_i |}{\Sigma | a_i d_i - b_i c_i |}$$

Example: An IFS Fern

Matrix elements:

i	ai	bi	ci	di	ei	fi
1	0.00	0.00	0.00	0.16	0.0	0.0
2	0.85	0.04	-0.04	0.85	0.0	1.6
3	0.20	-0.26	0.23	0.22	0.0	1.6
4	-0.15	0.28	0.26	0.24	0.0	0.44

Result after 2000 iterations:

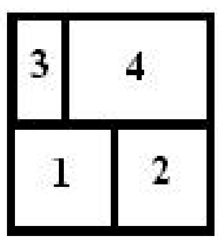
Result after 20,000 iterations:

Result after 200.000 iterations

Finding IFS for Arbitrary Images

- Collage Theorem (M. Barnsley)
  - Any image can be represented by union of contractive affine transformations of itself
  - So cover the image with reduced replicas of itself
    - a collage
  - Find transformation for each replica --> Mi
  - Process can be automated
  - Can be used in image compression

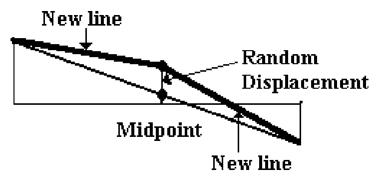
An IFS Square (one way)



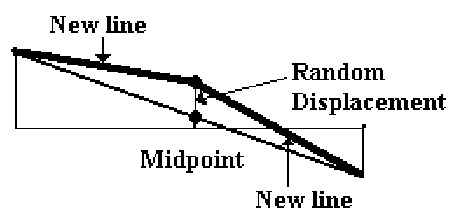
M1 = (0.5, 0, 0, 0.5, 0, 0), P1 = .25
M2 = (0.5, 0, 0, 0.5, x/2, 0), P2 = .25
M3 = (0.25, 0, 0, 0.5, 0, y/2), P3 = .125
M4 = (0.75, 0, 0, 0.5, x/4, y/2), P4 = .375

Random Midpoint Displacement

- Good for mountain silhouettes
- Recursive subdivision
- Start with a line segment
- Find midpoint (xm,ym)
- Displace ym by a random amount proportional to current length
- Repeat with each subdivision until sufficiently detailed
  - Repeat until we get to individual pixels
  - Store computed values of y in an array y[]

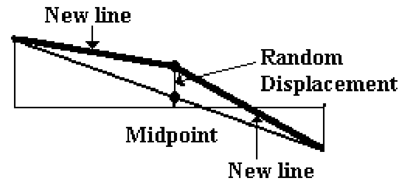


- Start endpoint coordinates: (x1,y1), (x2,y2)
- Assume we have a recursive procedure fracline(a,b)
  - Computes displaced midpoint line from x=a to x=b
  - Calls itself for each half of line
  - Repeat until y values for all pixels between endpoints are computed

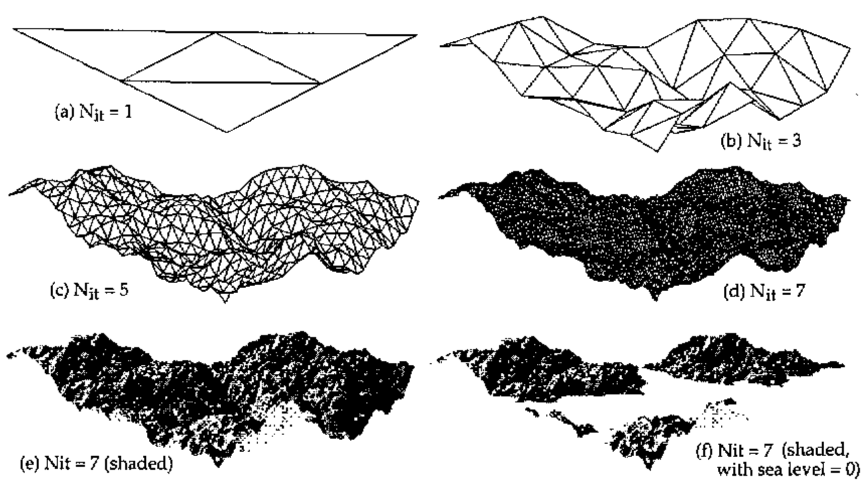


```
int y[SCREEN_WIDTH);
float rug = 0.5;              // ruggedness factor
y[x1] = y1;  y[x2] = y2;    // line endpoints
fracline (x1,x2);             // fills y array values
for (x=x1; x<=x2; x++)
  SetPixel(x,y[x]);

fracline (a,b)
   {   if ((b-a) > 1)
       {  xmid = (a+b)/2;
          y[xmid] = (y[a]+y[b])/2 + rug*(b-a)*rand();
          fracline (a, xmid); fracline (xmid, b); }
   }
```



- Generalize to triangular surfaces in 3D
- Displace each triangle edge midpoint randomly in z
- --> Neat mountains!



Drawing Trees With Recursive Subdivision

- A tree is a recursive structure
  - Each node is a new tree
- Draw trunk (first branch)
- Draw new branches from end of parent branch
  - Each new branch length reduced by a factor f
  - Each new branch goes off at an angle alpha with respect to parent branch
  - Recursive function branch(n,x,y,a,alpha)
    - n=level of recursion, x,y = endpoint of current branch, a = length of current branch, alpha = current branch angle

Plasmas

- Extension of random midpoint displacement
- Works with colors
- Great for generating clouds
- Easily generalized to give mountains

Plasma-generating Algorithm

Set screen black

Set current rectangle to entire screen

Set each corner pixel of current rectangle to a random color

For each edge of current rectangle

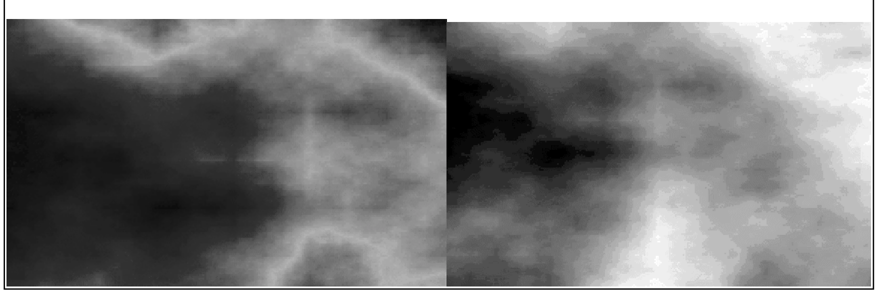
Compute color of midpoint P between edge's corner pixels by:

1. Pick a random color C
2. Compute weighting factor W proportional to distance between corner & P
3. Set midpoint color to average of two corner colors and the color C weighted by W

Set center of current rectangle to average of 4 edge midpoint colors

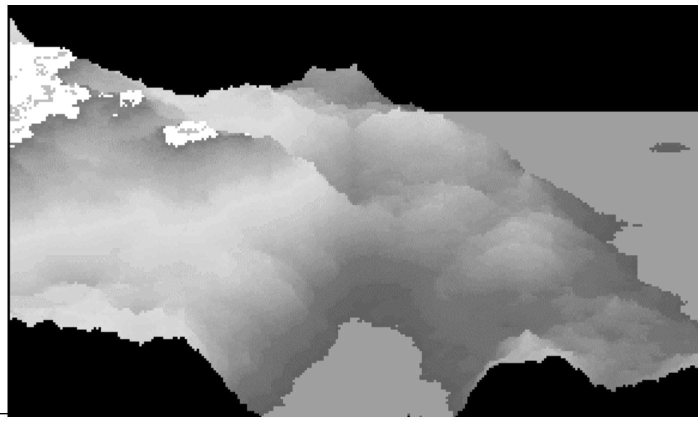
Repeat recursively for each new rectangle determined by corner pixel and center pixel until all pixels are colored

- Key idea--at beginning, distances are large
  - So color of center pixel is mostly random
  - But as rectangles become smaller,
    random contribution is less...
    while neighbor pixel contribution is greater
  - So close points have similar colors
    - Like clouds



Converting a Plasma to a Mountain

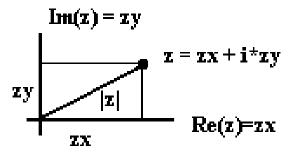
– Treat color code of each point as a height
– Plot the resulting surface
– (A cloud is a color-coded map of a mountain!)



Escape-Time Algorithms for Generating Fractals

- Give iterative rule for generating points in the complex plane
- Use "seed" points & determine if “orbit” of points generated by iterative rule is finite or escapes to infinity
- Map real (x) and imaginary (y) parts of each seed point to a pixel on screen
- Boundary between seed points whose orbits escape and those whose orbits do not escape is often a very complex fractal

Example: Mandelbrot Set



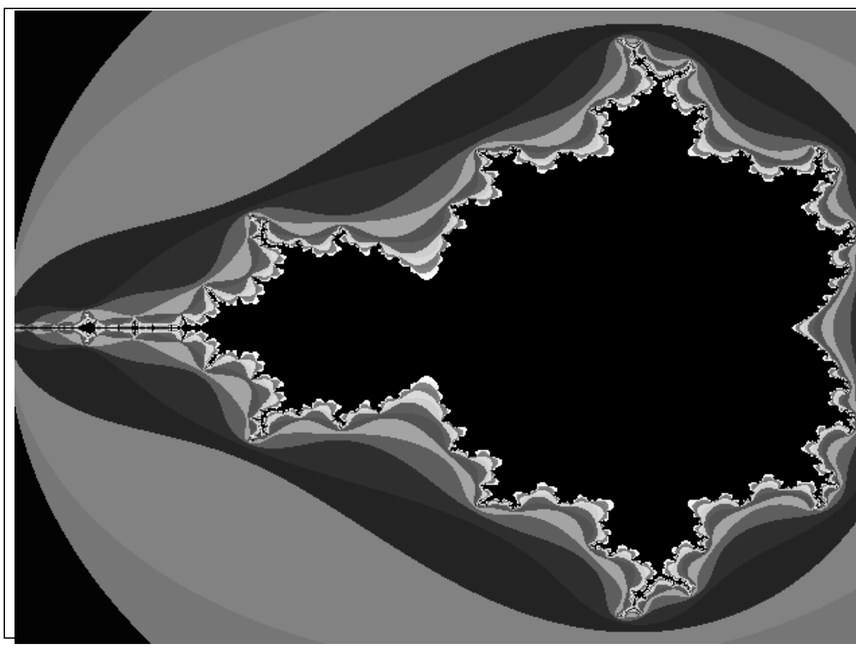
- Iteration rule: $z = z^2 + c$
- c is the seed point: c = cx + i*cy
- z = zx + i*zy is each new complex point generated
- Start out with z = (0,0)
- By definition $z^2 = (zx^2 - zy^2, 2*zx*zy)$
- Square of radius of orbit: $|z|^2 = zx^2 + zy^2$
- If |z| > 2, orbit will escape to infinity (can be shown)
- Area of complex plane containing Mandelbrot set:

  -2 < cx < 1.5 and -1.5 < cy < 0.5

Mandelbrot Set Algorithm

- Simple algorithm to generate image of Mandelbrot set
- Points in Set are painted black
- Points outside set are painted white
- Can be generalized to paint in colors
  - Depending on how quickly outside points escape to infinity

```
Set N to some large maximum number of iterations
For y = 0 to SCREEN_HEIGHT
   For x = 0 to SCREEN_WIDTH
      Map (x,y) to (cx,cy)  // inverse 2D viewing transformation
      zx = 0; zy = 0; count = 0;
      While ( (zx*zx + zy*zy < 4) && (count < N) )
         count++;
         temp = zx*zx - zy*zy + cx;  // real part of new z
         zy = 2*zx*zy + cy;          // imaginary part of new z
         zx = temp;
      If (count < N)
         Setpixel(x,y,white);    // orbit escaped to infinity
      Else
         Setpixel(x,y,black);    // orbit did not escape in N iterations
```



Grammar-Based Systems (Lindemayer, L-Systems)

- Objects represented by strings of letters
  - Need an “Alphabet”
    - used to compose strings
  - Need an initial word (“Axiom”)
    - successive generations of string derived from it
- “Productions” specify how new generations of objects are obtained
  - Give rewriting rules
    - applied in parallel to each letter in string

L-Systems in Computer Graphics

- Interpret each letter as a movement on screen (turtle graphics)
- Example alphabet with interpretation:

  F: Go forward (trace a line)

  +: Turn left by a given angle

  - : Turn right by a given angle

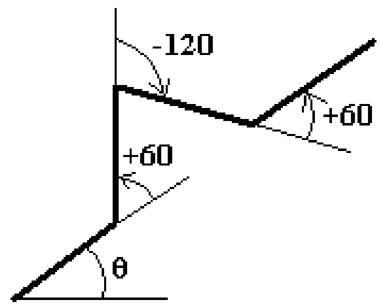
  many other possible movements

L-System for a Koch Curve

Alphabet:
- F, +, -
  - Forward, turn +/-
- Take angle as 60

Axiom:
- F

Production:
- F -> F + F - - F + F



Deriving the System

F -> F + F - - F + F

Next iteration

- (F+F--F+F) + (F+F--F+F) - - (F+F--F+F) + (F+F--F+F)

Successive iterations generate the Koch Curve

L-Systems can be extended in many ways

- Bracketed L-Systems
  - Good for modeling plants
  - Anything inside brackets is a branch
  - “[” means push onto stack (start branch)
  - “]” means pop from stack (end branch)
- Stochastic L-Systems
  - Apply productions probabilistically
- Lots of other variations

Particle Systems

- Collections of particles that evolve over time
- Used to model systems whose time behavior is unpredictable
- Evolution determined by applying laws of physics to each particle
- Probabilistic effects easily included

Particles can:

- Be born and die
- Generate new particles
- Change their attributes
  - color, mass, etc.
- Move according to specified laws of motion
- Interact with their environment
- Interact with each other

Particles can model:

- Fire
- Clouds
- Fog
- Explosions
- Moving water
- Flocking birds
- Lots of other systems

End of Course Stuff

- Final Exam
  - Open books & notes
  - Thursday, May 13, 2010
  - 11:00 A.M-1:00 P.M.
  - FA-212

Final Exam Topics

- 3D Geometric Transformations
  - Translation; Rotation about x, y, z axes; Scaling
- The 3D Modeling/Rendering Pipeline
  - 3D Polygon Mesh Model Data Structures (Points, Polygon lists)
  - 3D Viewing Transformation (4-parameter viewing setup)
  - Projection Transformations (perspective, parallel)
  - Window to Viewport Transformation
- 3D Modeling and Rendering with OpenGL
- Back-Face Culling
- Z-Buffer Hidden Surface Removal Algorithm
- Illumination and Reflection (ambient, diffuse, specular)
- The Phong Illumination/Reflection Model
- Flat Shading
- Interpolated Shading (Gouraud)
- Ray Tracing & Texture Mapping
- Fractals