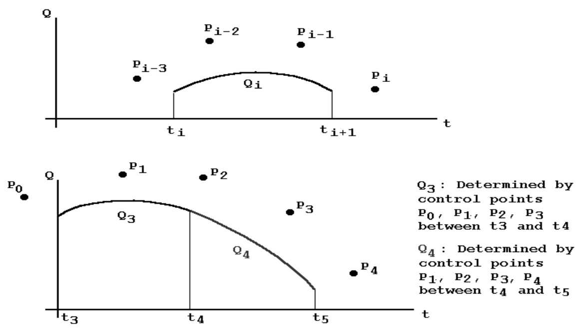
B-Spline Polynomials

B-Spline Polynomials

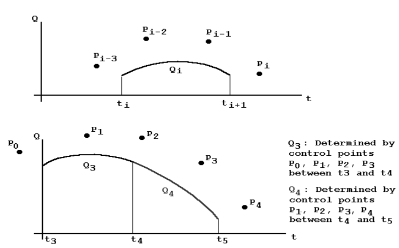
- Draftman's spline
  - Flexible metal strip used to lay out object surfaces
  - If not stressed too much, get level-2 (2^{nd} derivative) continuity curve
- Want local control
- Smoother curves
- B-spline curves:
  - Segmented approximating curve
  - 4 control points affect each segment
    - Local control
  - Level-2 continuity everywhere
    - Very smooth

Cubic B-Spline Polynomial Curves

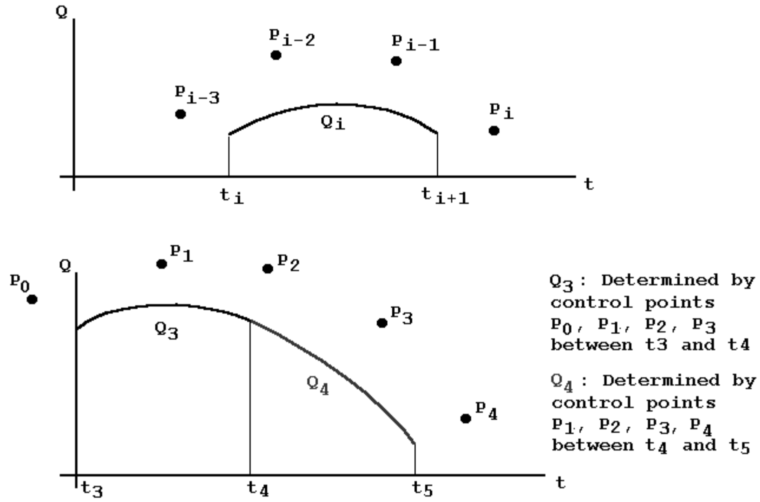
- Approximate m+1 control points Pi (i=0,1,2,…,m) with a curve consisting of m-2 cubic polynomial curve <u>segments</u> Q_i (i=3,4,...m), m>=3
- Each Q_i defined in terms of:
  - parameter t: $t_i <= t <= t_{i+1}$ and by four of the m+1 control points



- Segment Q_i determined by control points: P_{i-3}, P_{i-2}, P_{i-1}, P_i between t_i and t_{i+1}
- Q_i begins at $t = t_i$ and ends at $t = t_{I+1,}$
- Q_{i+1} joins Q_i at t_{i+1}
  - Join point called a knot.
- For example
  - First segment is Q_3, begins at t_3, ends at t_4
  - Is determined by control points P0,P1,P2,P3
- Each segment is affected by only 4 control points
- Each control point affects at most 4 curve segments



Uniform Cubic B-Spline Curves



Uniform Cubic B-Splines

- A special case where we assume that:
- $t_{i+1} = t_i + 1$
- Polynomial equation for segment Qi:
- $Qi(t) = a*(t-ti)^3 + b*(t-ti)^2 + c*(t-ti) + d$, $ti <= t <= ti+1$
- Take independent variable as t-ti
  - Will vary from 0 to 1 for any interval

- Need to get polynomial coefficients (a,b,c,d)
  - from control points
- Find a "B-Spline Basis Matrix"
  - as for Bezier curves
  - but must do computation for each interval

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = M_{BS} * \begin{bmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{bmatrix}$$

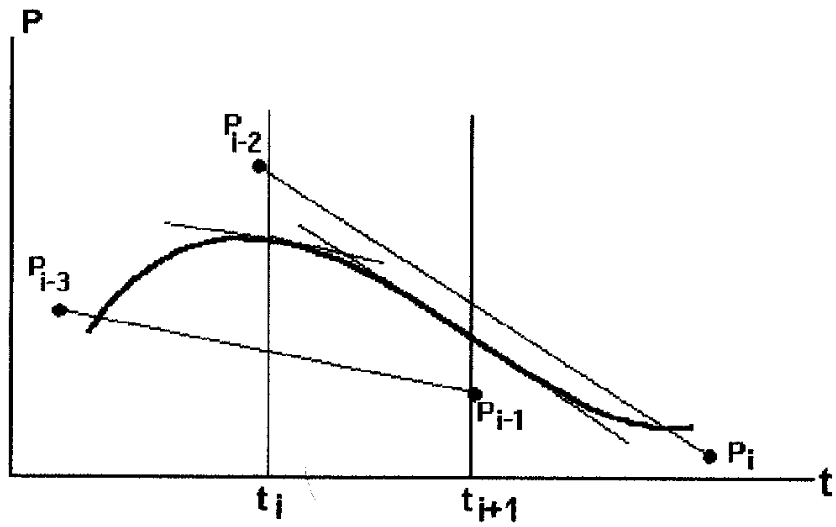
- M_{BS} is the desired matrix

B-Spline Continuity Conditions

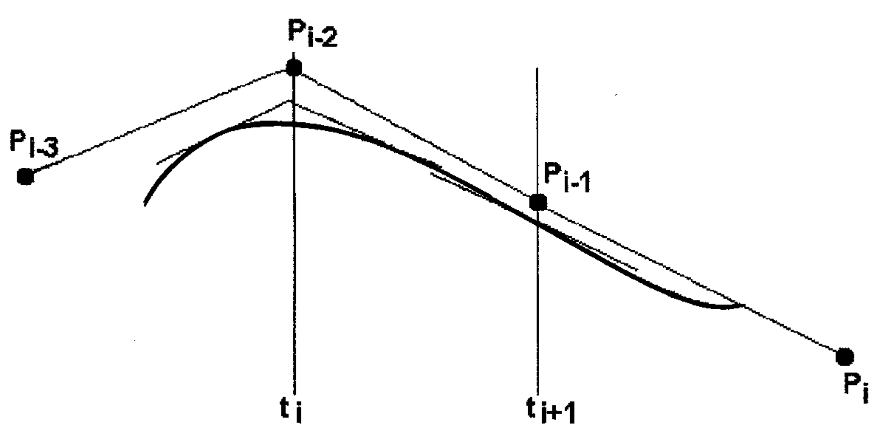
- Conditions on 1st & 2nd derivatives:

1. dQ_i/dt (at $t=t_i$) = slope of line segment joining P_{i-3} and P_{i-1}
2. dQ_i/dt (at $t=t_{i+1}$) = slope of line segment joining P_{i-2} and P_i
3. $(dQ_i/dt)'$ ($t=t_i$) = rate of change in slope at t_i :
   (slope of $[P_{i-3}-P_{i-2}]$ - slope of $[P_{i-2}-P_i]$) / Δt
4. $(dQi/dt)'$ ($t=t_{i+1}$) = rate of change in slope at t_{i+1} :
   (slope of $[P_{i-2}-P_{i-1}]$ - slope of $[P_{i-1}-P_i]$) / Δt

Continuity Conditions 1 and 2



Continuity Conditions 3 and 4



$Qi = a*(t-ti)^3 + b*(t-ti)^2 + c*(t-ti) + d$

$dQi/dt = 3*a*(t-ti)^2 + 2*b*(t-ti) + c$

$(dQi/dt)' = 6a*(t-ti) + 2*b$

- Condition 1: $c = (P_{i-1} - P_{i-3})/2$
- Condition 2: $3*a + 2*b + c = (P_i - P_{i-2})/2$
- Condition 3: $2*b = ((P_{i-1} - P_{i-2}) - (P_{i-2} - P_{i-3})) / 1$
- Condition 4: $6*a + 2*b = ((P_i-P_{i-1})-(P_{i-1}-P_{i-2})) / 1$
- These four equations are not independent
  - Solving gives only a, b, c, but not d

Solution

$a = (1/6) * (-P_{i-3} + 3*P_{i-2} - 3*P_{i-1} + P_i)$

$b = (1/6) * (3*P_{i-3} - 6*P_{i-2} + 3*P_{i-1})$

$c = (1/6) * (-3*P_{i-3} + 3*P_{i-1})$

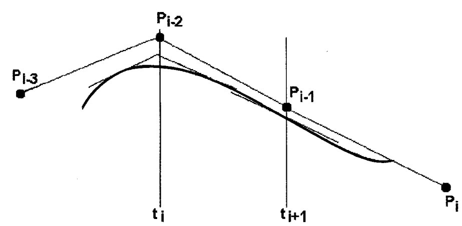
Need another condition to get d

- Choose the following condition:

  Q (at t=ti) = (1/6) * (P $_{i-3}$ + 4*P $_{i-2}$ + P $_{i-1}$)

  – i.e., control point at ti (P_{i-2}) pulls 4 times as hard at t=ti as control points on either side of ti
- Substitute in polynomial equation -->

  d = (1/6) * (P $_{i-3}$ + 4*P $_{i-2}$ + P $_{i-1}$)



Uniform Cubic B-Spline Coefficient Matrix Equation

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = (1/6)*\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} * \begin{bmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{bmatrix}$$

Could also be written in terms of blending functions

$$Q_i(t) = \sum_{j=0}^{3} B_{i-j,4}(t) * P_{i-j}$$

$B_{i-3,4} = 1/6 * (1-t)^3$
$B_{i-2,4} = 1/6 * (3t^3 - 6t^2 + 4)$
$B_{i-1,4} = 1/6 * (-3t^3 + 3t^2 - 3t + 1)$
$B_{i,4} = 1/6 * t^3$

See Foley & Van Dam

Plotting Uniform Cubic B-Splines

- Given m+1 control points P0,P1,P2,...Pm
  - (Recall that each has an x and y coordinate)
    - i.e., P0 --> x0 and y0, etc.
- The following is a "brute force" algorithm to plot the curve
  - delta is a very small increment (e.g., 0.05)

Brute Force Algorithm

```
For (i=3 to m)
    Compute ax,bx,cx,dx and ay,by,cy,dy
         from control points i-3, i-2, i-1, i
    For (t=0; t<=1; t+=delta)
      x = ax*t^3 + bx*t^2 + cx*t + dx
      y = ay*t^3 + by*t^2 + cy*t + dy
      If (t==0)
        MoveTo(x,y)
      Else
        LineTo(x,y)
```

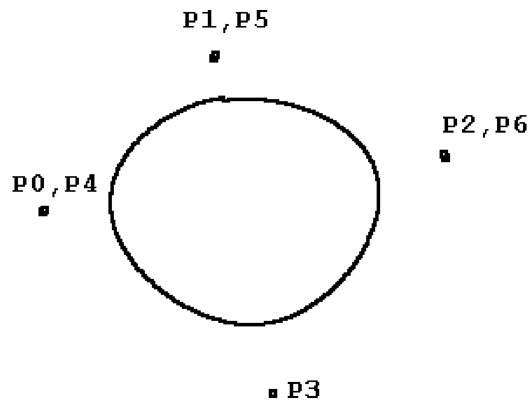
- To increase performance, use forward differences

Closed Cubic B-Splines

Make last 3 control points coincide with 1st 3

0 <--> m-2, 1 <--> m-1, 2 <--> m

Example: m=6



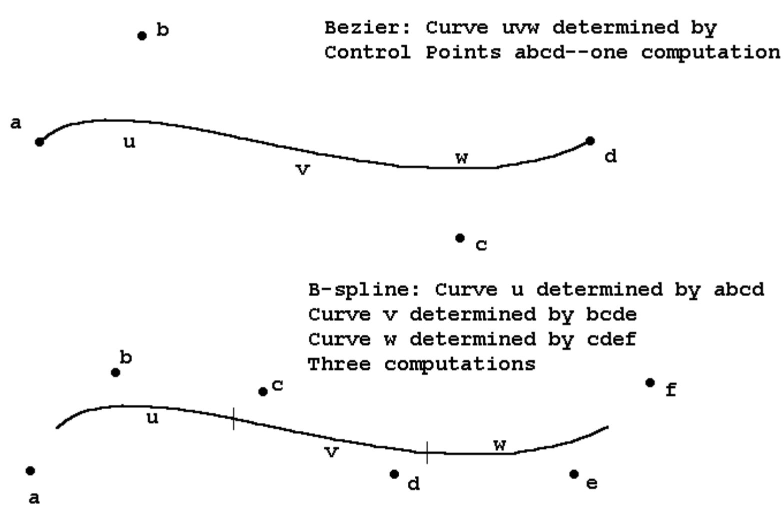
Forcing Interpolation

- Reproduce a control point three times
- Curve will then go through that point

Properties of Uniform B-Splines

1. Local Control
   – Each segment determined by only 4 control points
2. Approximates control points; doesn't interpolate
   (However it will interpolate triplicated control points)
3. Lies inside convex hull of control points
   – Each segment lies inside convex hull of its 4 control points
4. Invariant under affine transformations
5. Very smooth
   – Level-2 continuity everywhere
6. More computations required than for "equivalent" Bezier curve

Bezier vs. B-Spline Curves



Non-uniform Cubic B-Splines

- Greater variety of curve shapes
- Can have cusps and discontinuities
- Intervals between successive knots varies
- Knot values must be specified

  $t_0, t_1, t_2, t_3, t_4, \ldots, t_{m-2}$

NON-UNIFORM CUBIC B-SPLINES

Variable size intervals between successive knot values

Must specify knot values --> the knot vector,
a non-decreasing sequence
e.g., (0,0,0,0,1,1,2,3,4,4,....)

Can have multiple knots

The curve segment Q_i is determined by control points: P_{i-3}, P_{i-2}, P_{i-1}, P_i

and by blending functions: $B_{i-3,4}(t)$, $B_{i-2,4}(t)$, $B_{i-1,4}(t)$, $B_{i,4}(t)$

[4 = the order (degree-3 plus 1) of the polynomials]

is given by:

$$Q_i(t) = P_{i-3} B_{i-3,4}(t) + P_{i-2} B_{i-2,4}(t) + P_{i-1} B_{i-1,4}(t) + P_i B_{i,4}(t)$$

$3 <= i <= m$, $t_i <= t < t_{i+1}$ defined between t_3 and t_{m+1}

If $t_i = t_{i+1}$ then the curve segment Q_i degenerates to a point.

The Blending functions B(t) are defined recursively:

$$B_{i,1}(t) = \begin{cases} 1, & t_i \le t < t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

$$B_{i,2}(t) = \frac{t - t_i}{t_{i+1} - t_i} B_{i,1}(t) + \frac{t_{i+2} - t}{t_{i+2} - t_{i+1}} B_{i+1,1}(t)$$

$$B_{i,3}(t) = \frac{t - t_i}{t_{i+2} - t_i} B_{i,2}(t) + \frac{t_{i+3} - t}{t_{i+3} - t_{i+1}} B_{i+1,2}(t)$$

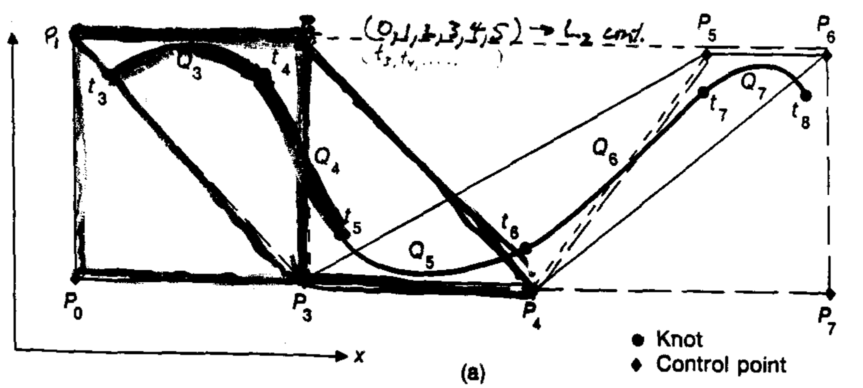
$$B_{i,4}(t) = \frac{t - t_i}{t_{i+3} - t_i} B_{i,3}(t) + \frac{t_{i+4} - t}{t_{i+4} - t_{i+1}} B_{i+1,3}(t)$$

In these equations, 0/0 is defined to be equal to 0.

Case A (Level-2 Continuity)

- Knot vector: (0,1,2,3,4,5,...)
  - Just our friend the uniform B-spline
- Q3 determined by P0, P1, P2, P3
- Q4 determined by P1, P2, P3, P4
- Q3 and Q4 share control points P1,P2,P3
  - Three constraints ==> L0, L1, L2 continuity

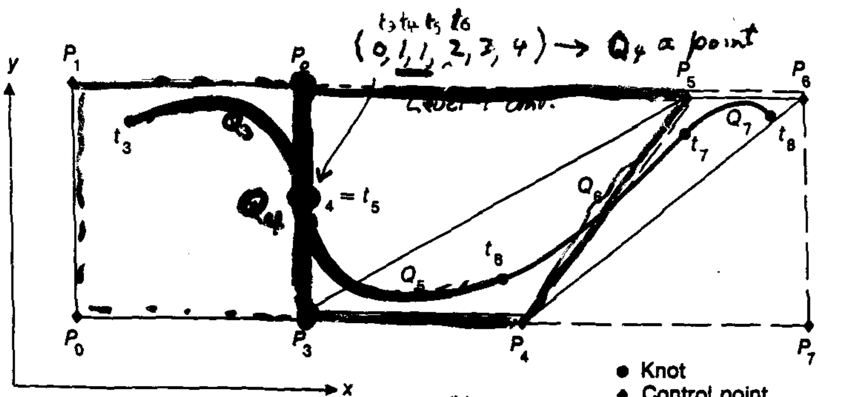
Case A (Level-2 Continuity)



Case B (Level-1 Continuity)

- Knot vector: (0,1,1,2,3,4,...)
- Segment Q4 becomes a point
  - (since t4 = t5)
- Q3 determined by P0, P1, P2, P3
- Q5 determined by P2, P3, P4, P5
- So Q4 must lie on line connecting P2 & P3
- Q3 and Q5 share control points P2 & P3
  - Two constraints ==> L0, L1 continuity

Case B (Level-1 Continuity)

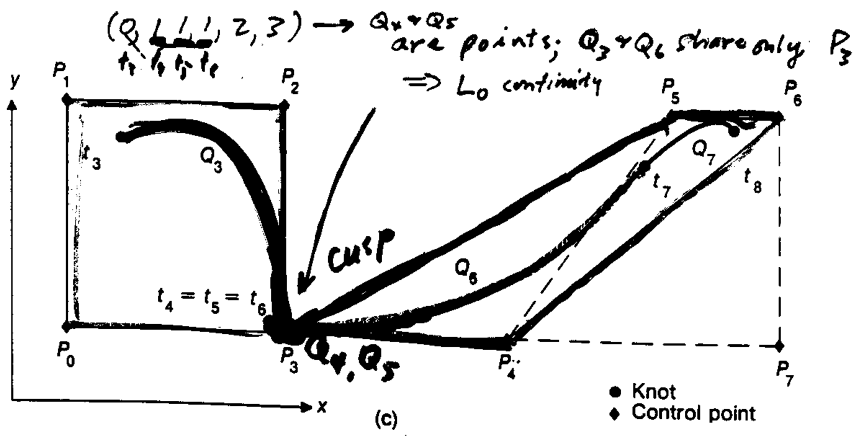


(b)

Case C (Level-0 continuity)

- Knot vector: (0,1,1,1,2,3,...)
- Q4 and Q5 become points
  - (since t4=t5=t6)
- Q3 determined by P0, P1, P2, P3
- Q6 determined by P3, P4, P5, P6
- So Q4/Q5 must lie on control Point P3
  - (interpolates it)
- Q3 and Q6 share control point P3
  - One constraint ==> L0 continuity

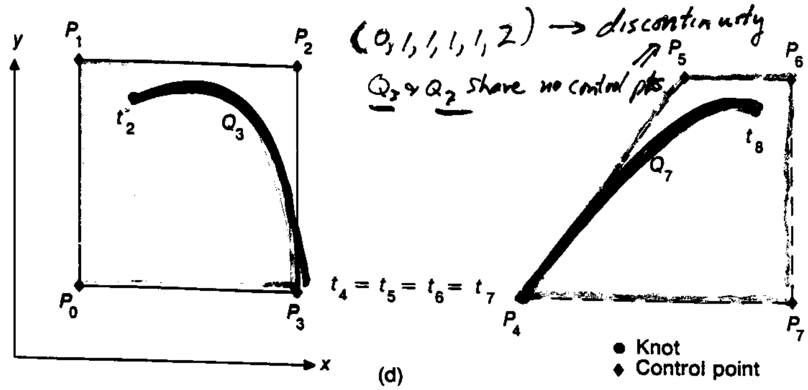
Case C (Level-0 continuity)



Case D (No Continuity-Gaps)

- Knot vector: (0,1,1,1,1,2,...)
- Q4, Q5, Q6 become points
  - (since t4=t5=t6=t7)
- Q3 determined by P0, P1, P2, P3
- Q7 determined by P4, P5, P6, P7
- There is no overlap
- Q3 and Q7 share no control points
  - No constraints ==> discontinuity

Case D (No Continuity)



(d)

3-D Graphics

Overview of 3-D Computer Graphics

- Display image of real or imagined 3-D scene on a 2-D screen

Introduction to 3-D Graphics

- Modeling and Rendering
- Polygon Mesh Models
- Bicubic Patch Models
- Solid Models

Problem # 1: Modeling

- Representing objects in 3-D space
- First need to represent points
- Use a 3-D coordinate system, e.g.:
  - Cartesian: (x, y, z)
  - Spherical: (rho, theta, phi)
  - Cylindrical: (r, theta, z)

Conversions

- Spherical to Cartesian

  $x = \rho * \sin(\phi) * \cos(\theta)$

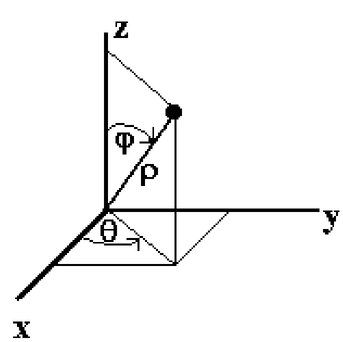
  $y = \rho * \sin(\phi) * \sin(\theta)$

  $z = \rho * \cos(\phi)$

RH Coord System

Could be LH

  Viewing system



Types of 3-D Models

- 1. Boundary Representation (B-Rep)
  - Surface descriptions
  - Two common ones:
    - A. Polygonal
    - B. Bicubic parametric surface patches
- 2. Solid Representation
  - Solid modeling

Polygonal Models

- Object surfaces approximated by a mesh of planar polygons

Scene -->

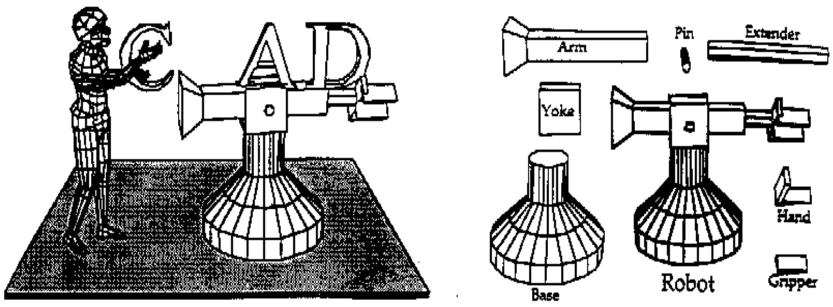
Objects -->

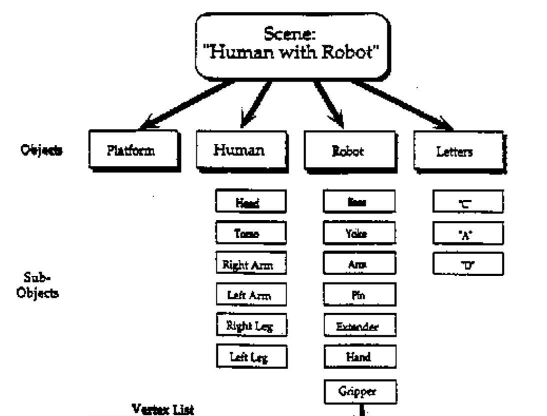
Sub-objects -->

Polygons -->

Vertices (points)

Polygon Mesh Model Example Scene





Data Structure

Vertex List

Vertex #	Coordinates
1	x1,y1,z1
2	x2,y2,z2
3	x3,y3,z3
4	x4,y4,z4
5	x5,y5,z5
6	x6,y6,z6
7	x7,y7,z7
8	x8,y8,z8

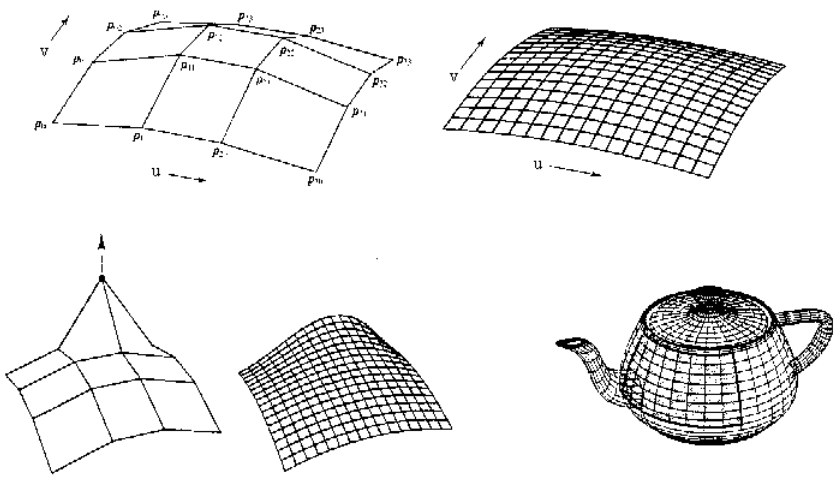
Polygon List

Gripper	
Polygon #	Vertex List
1	V1,V2,V3,V4
2	V1,V4,V8,V5
3	V5,V6,V2,V1
4	V2,V6,V7,V3
5	V4,V3,V7,V8
6	V5,V8,V7,V6

Bicubic Parametric Surface Patches

- Objects represented by nets of elements called surface patches
  - Polynomials in two parametric variables
  - Usually cubic
    - Bezier surface patches
    - B-Spline surface patches

Bicubic Parametric Surface Patches

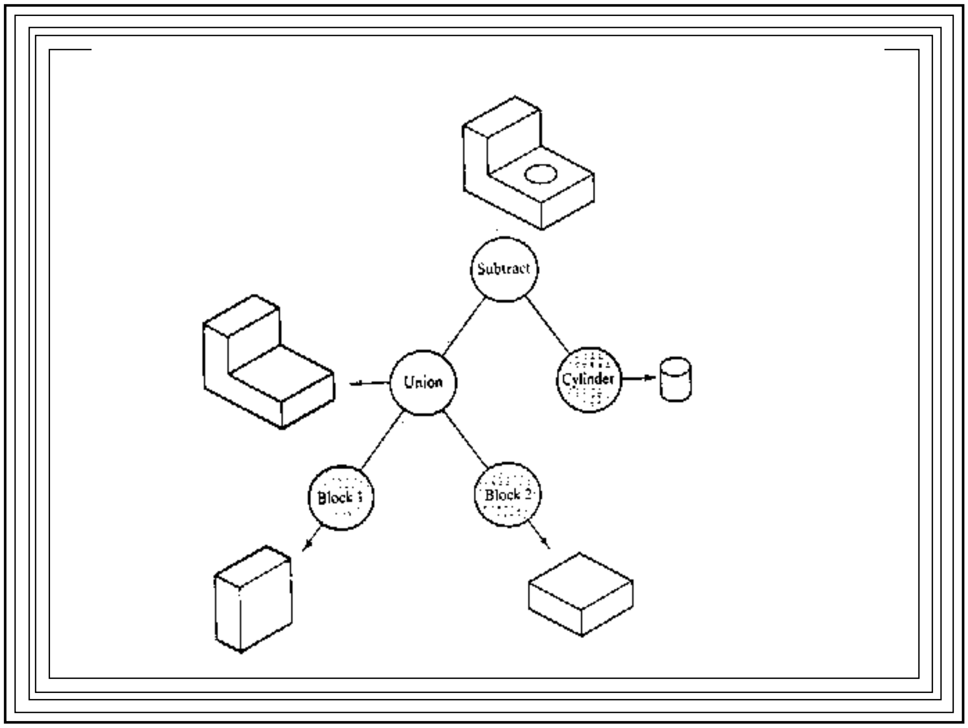


Solid Representation-- Solid Modeling

- Objects represented exactly by combinations of elementary solid objects
  - e.g., spheres, cylinders, boxes, etc
  - Called geometric primitives

Constructive Solid Geometry (CSG)

- Complex objects built up by combining geometric primitives using Boolean set operations
  - union, intersection, difference
- and linear transformations
- Object stored as a tree
  - Leaves contain primitives
  - Nodes store set operators or transformations



Subtract
Union
Cylinder
Block 1
Block 2